DARRÉ'S ELEMENTS

OF

PLANE AND SOLID

GEOMETRY.

REVISED AND IMPROVED

BY THE

REV. F. LENNON,

PROFESSOR OF MATHEMATICS AND NATURAL PHILOSOPHY
IN MAYNOOTH COLLEGE.

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PREFACE.

THESE Elements of Geometry were compiled, about seventy years ago, by Abbé Darré, a French refugee, for the use of his pupils in Maynooth College. They were subsequently enlarged and very much improved by the late lamented Doctor Callan, and were always used by him as a text-book in his annual course of Mathematical Lectures.

The present edition, though for the most part re-written, differs not so much in substance as in form from those that have preceded it. A new theorem, or a more comprehensive form of an old one, has been introduced occasionally; but, except in a few instances, the order has been preserved. With a view to greater accuracy, many of the definitions and some of the demonstrations have been considerably modified; and, instead of the *method of indivisibles*, that of *limits*, or some equivalent method, has been substituted throughout. In most of the changes that have been made, I adopted as models the best hand-books on the subject at present in use in France and Germany: since Euclid, however, notwithstanding its many defects, is still retained in nearly all the schools and colleges in these countries, I deemed it advisable

to borrow from it a few problems and one or two theorems, not given in former editions of this treatise; and I think it will be found that every proposition of importance usually read in the first six and the eleventh and twelfth books of Euclid is contained, in substance at least, in the text or exercises.

The introduction of 'proportion,' with all the difficulties of 'incommensurables,' so early as pp. 10 and 60, will appear anomalous to persons not acquainted with the Maynooth mathematical course; but it must be borne in mind that the Continental practice of reading Algebra before Geometry, initiated by Darré at the end of the last century, is still kept up, and will continue, I trust, for years to come.

Slight imperfections in a few of the diagrams are to be ascribed to the hurry in which I was obliged to make some of the drawings, and not to the engraver, who has executed his task with the utmost exactness and despatch.

The treatises on Trigonometry which in former editions accompanied the Elements of Geometry are omitted in the present, but will be ready in a separate form, it is hoped, in a few months.

F. L.

MAYNOOTH COLLEGE,
Sept. 20, 1872.

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Page 3, line 22	from top,	*for* A *read* An.
,, 48, ,, 25	,,	*for* anlges *read* angles.
,, 56, ,, 23	,,	*for* arc BAG *read* the arc BAG.
,, 74, ,, 29	,,	*for* sides *read* lines.
,, 87, ,, 27 and 33	,,	*for* (i) *read* (1).
,, 106, ,, 17	,,	*for* best *read* least.
,, 111, Fig. 108, *draw the line* HK.		

ELEMENTS OF GEOMETRY.

PART I.

PRELIMINARY DEFINITIONS AND AXIOMS.

GEOMETRY is the science of extension and of its measurement.

The extension of a body is the portion of space which the body fills. It comprises three dimensions, length, breadth, and depth or thickness. But, although these three dimensions are always found together in material objects, yet, any one may be considered apart from the others, and two of them may be considered without attending to the third. When we look at a block of wood or marble, we can easily think of the space which it occupies, without thinking of the kind of substance of which the block is composed. This enables us to realise what is meant by a *geometrical solid*, or *volume*. If, now, by a process of mental abstraction, we remove from our thoughts all idea of thickness, and attend only to the length and breadth, we form a conception of *surface*. By abstracting from the breadth, and considering the length of the surface only, we get the idea of a *line*. Finally, we conceive a *point*, when we abstract from the length of a line, and consider only the *position* of either extremity. A *geometrical point*, therefore, has neither length, breadth, nor thickness. It differs from a *physical point*, such as a dot made on paper, because the latter has a very small magnitude.

And, in like manner, a *geometrical line*, or *surface*, differs from a *physical line*, or *surface*. Hence diagrams, in which physical lines only can be used, are necessarily imperfect.

The result of the preceding is briefly expressed in the following definitions:—

Def. I. A *solid* is that which has length, breadth and thickness.

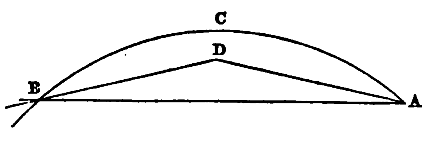
Def. II. A *surface* is that which has length and breadth, but no thickness.

Def. III. A *line* is length, without breadth or thickness.

Def. IV. The extremity of a line, or any of its intersections with another line, is called a *point*.

Besides the foregoing *analytic* method, there is another called the *synthetic*, by which the mind is greatly assisted in forming correct notions of a *geometrical line*, *surface*, &c. Thus:—Holding a ball in the hands, we can easily conceive the *position* in space of its centre, and this position remains unchanged even when the ball is removed. Position thus regarded is a *point*. Let this point be conceived as moving forward, and its path is a *line*—*a straight line*, as AB (Fig. 1), when the point moves so as to deflect neither to the right nor to the left, neither up nor down; that is, when it moves always in the *same* direction—a *curved line*, as ACB, when the direction of its motion is constantly changing; and a *broken line*, as ADB, when it moves for a time in one direction, and then in another.

Fig. 1.



If a line be conceived as moving in a direction different from that of its length, it will generate a *surface*. Lastly, when a surface moves in any other direction than that of its length and breadth, it generates a *solid*. Hence,

Def. V. *A straight line* is a line formed by a point constantly moving in the same direction.

1. Between two given points, such as A and B (Fig. 1), there can be drawn only one straight line. Because, since the point necessary to generate a second straight line must

move always in the same direction, and must pass through A and B, it cannot be found outside the straight line AB.

2. Hence it follows that two straight lines cannot have more than *one* common point of meeting; and, when *two* points are common to two straight lines, the lines must coincide.

3. It may be assumed, as self-evident, that a straight line is the shortest line which can be drawn between two points; for, if a generating point start from A (Fig. 1), and move in each case at the same rate, clearly, it will reach B sooner by going along the straight line AB, than by going along the curved line ACB, or the broken line ADB.

4. A straight line is the proper measure of the *distance* between one point and another; because, it is the *shortest* line between them, and of invariable length, when the points are fixed.

5. When any two points in a straight line are determined, the *position* of the line is determined; and no ambiguity can arise, as to the line intended, when any two points in it are mentioned. Hence, to designate a straight line, we shall use two letters, placed at different points of the line.

6. A indefinite number of curved lines may be drawn between two given points; for a curved line is formed by a point which constantly changes the direction of its motion, and the change may take place in an indefinite number of ways. Three letters at least, therefore, are necessary to designate a curved line when two of them are placed at its points of intersection with another line.

In what follows, when a line is mentioned without any qualifying word, a straight line is to be understood.

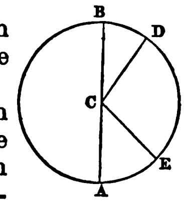
Def. VI. When every point of a straight line, in every direction in which it can be placed along a surface, is in contact with the surface, that surface is called a *plane.*

A plane may be conceived as formed by the motion of a *straight* line in any one direction different from that of the length of the line.

Def. VII. A *circle* is a plane surface contained within a curved line, called its *circumference*, every point of which is equidistant from a certain inside point of that surface (Fig. 2).

The circumference of a circle may be considered as formed by a point which so moves in a plane that its distance from a fixed point in that plane is always the same.

Fig. 2.



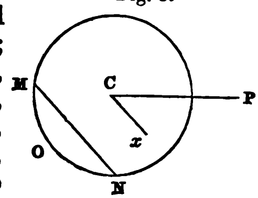
The fixed point from which every point in the circumference is equidistant is called the *centre.*

Def. VIII. A straight line passing through the centre, and terminated on both sides by the circumference, is a *diameter;* and a line from the centre to the circumference is called a *radius.*

1. The diameter is equal to two radii, and all radii of the same circle are equal, because they measure equal distances. Thus, AB (Fig. 2), is equal to CA and CB, and also to CD and CE, when joined together, so as to form one straight line; that is, it is equal to their *sum.*

2. The circumference of a circle passes through all the points in the plane of the circle, which are at a distance from the centre equal to the radius; for a line from the centre to a point such as P (Fig. 3), outside the circumference, must cut the latter, and is, therefore, greater than a radius; and a line from the centre to a point, such as x inside the circumference, must be produced to meet it, and is, therefore, less than a radius. Hence, neither P nor x can be at a distance from the centre equal to the radius.

Fig. 3.



Def. IX. An *arc* is any part of the circumference; and a straight line, which joins the two extremities of any arc, is called the *chord* of that arc.

Thus, the straight line MN (Fig. 3), is the chord of the arc MON. The same chord is common to two arcs; but, unless the contrary be expressed, it is always to be understood that the less arc is intended.

Def. X. The space contained within two radii and their intermediate arc is a *sector;* and the space contained between a chord and its arc is called a *segment.*

Def. XI. Unequal circles, which are in the same plane, and have a common centre, are called *concentric* circles.

Def. XII. A space entirely enclosed by lines, straight or curved, is a *figure*.

N. B.—The word circle is frequently used in the sense of circumference, but the context in all cases easily determines its meaning.

By an *axiom* in geometry is meant a statement whose truth is so obvious, that it does not require, nor admit of, strict demonstration.

Two statements already made (*Def.* V., 2, 3), are commonly regarded as axiomatic; and, for convenience sake, we shall quote them henceforward as such.

Axiom 1. Two straight lines cannot meet each other in more than one point.

Axiom 2. A straight line is the shortest line that can be drawn between any two points.

The common axioms of arithmetic and algebra are also frequently employed in geometry.

A *theorem* is a truth which is deduced by a process of reasoning from other known truths.

A *corollary* to a theorem is a truth which is easily inferred from the theorem.

A statement, in which something is proposed to be done, is called a *problem;* and the correct performance of the operation indicated is the *solution* of the problem. The method of solving each of the problems contained in the following *postulates* is so obvious, that the mind readily accepts the problems as solved.

POSTULATES.

1. A straight line may be drawn from one given point to another.
2. A straight line may be produced to any length.
3. A circle may be described with any point as centre, and any line as radius.
4. A line or figure may be conceived as transferred from

any one position to any other, its magnitude and form remaining unaltered.

The signs +, −, ×, ÷, =, >, <, $\sqrt{}$, &c., when used in the following treatise, without any expressed restriction or extension of meaning, are to be understood in the same sense as in algebra.

SECTION I.

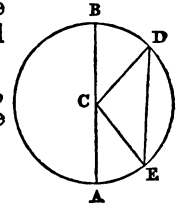
ON LINES AND ANGLES.

THEOREM.

1. *The diameter is the longest of all chords.*

Let AB (Fig. 4) be a diameter, and ED any chord which does not pass through the centre. Draw to the extremities of the chord the radii CE and CD.

Fig. 4.



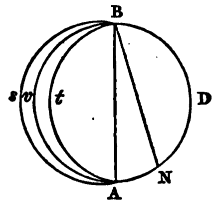
Then AB = CD + CE (*Def.* VIII. 1), and CD + CE > ED (Ax. 2). Therefore AB > ED.

THEOREM.

2. *A diameter bisects the circle and its circumference; and a chord which bisects the circle or its circumference is a diameter.*

1°. If AB be a diameter of the circle A*v*BD (Fig. 5), it will bisect the circle and its circumference. Let the figure ADB be conceived as turned over and superimposed on A*v*B, the diameter AB remaining fixed. Then must ADB coincide with A*v*B; for if the arc ADB take the position A*t*B, or the position A*s*B, since the centre is in AB, all the radii cannot be equal, which is contrary to the essence of a circle (*Def.* VIII. 1). Therefore the arc ADB must coincide

Fig. 5.



with the arc AvB, and consequently the figures ADB and AvB are equal.

2°. If AB be a chord which bisects the circle or its circumference, AB must be a diameter. If not, draw from B a line BN which passes through the centre, and by the preceding BN bisects the circle and its circumference. Hence, if AB bisects the *circle*, the figures BDN and BDA are each one half of it, and must therefore be equal; that is, BN must coincide with BA: and if AB bisects the *circumference*, the arc BDN = arc BDA—an evident impossibility unless the points N and A coincide, that is (*Def.* V. 2), unless BN coincides with BA, which is therefore a diameter.

Cor. i. *When a chord divides a circle into two unequal segments, the centre is in the greater.*

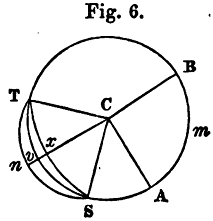
Because if from one extremity of the chord BN (Fig. 5), dividing the circle into two unequal segments, a line BA is drawn, which cuts as much off the larger segment BvN as, when added to the smaller segment BDN, makes it equal to a semicircle, that line, it is obvious, must pass through the greater segment, and, by the theorem, must also pass through the centre.

THEOREM.

3. *An arc taken at any part of the circumference coincides with the circumference when superimposed on it at any other part.*

Let AmB (Fig. 6) be any arc of the circle ABT, and it will coincide with the circumference if applied to it at any other part. Draw the radii AC and BC, and place the sector ACB on any other part of the circle, so that C may be at the centre. Then A and B must fall on the circumference (*Def.* VIII. 1). Let A and B coincide with T and S, and the arc AmB must coincide with the arc TvS. If not, let it fall inside on TxS, or outside on TnS. In the first case

Fig. 6.



$Cv > Cx$, which is absurd (*Def.* VIII. 1). In the second case $Cv < Cn$, which is also absurd (*Def.* VIII. 1).

Therefore the arc AmB must coincide with the arc TvS.

Cor. i. Hence the form or curvature of the circle is the same at every part of its circumference, and consequently the circumference of the circle is divisible into any number of equal parts, that is, into parts which coincide when any one is properly applied to any other.

Cor. ii. The chords of equal arcs are equal; because equal arcs coincide when superimposed on each other, and consequently their chords, or the straight lines joining their extremities, must coincide (*Def.* V. 2), and be equal to each other. The same is obviously true when the equal arcs are taken in *equal* circles, that is, in circles which have an equal radius.

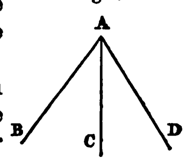
ANGLES.

4. *Def.* An *angle* is the divergence of two straight lines which meet each other.

The lines are called the *sides*, and the point of meeting, the *vertex* of the angle. An angle is designated by a single letter at the vertex, when the same vertex is not common to two or more angles. But if two or more angles have a common vertex, each is designated by three letters, one of which is at the vertex, and one at each of the sides, the letter at the vertex being placed between the other two. Thus the angles in Fig. 7 are named BAC, CAD, and BAD.

An angle may be conceived as generated by a straight line AD revolving round the point in which it cuts another straight line AB which is fixed.

Fig. 7.



It is obvious that the magnitude of an angle does not depend on the length of the sides, but only on the amount of opening or divergence of the lines which form it. When two angles BAC and CAD are placed as in Fig. 7, the angle BAD is called their *sum;* and CAD is the *difference* of the angles BAD and BAC.

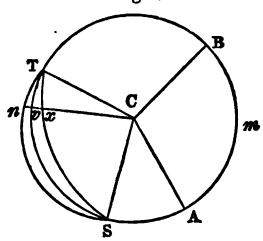
Two angles are said to be *equal* when, one being applied

to the other, the vertex and sides of the one can be made to coincide with the vertex and sides of the other.

THEOREM.

5. *Sectors having equal angles in the same circle, or in equal circles, stand on equal arcs, and reciprocally.*

Fig. 7*a*.



1°. Let ACB and TCS (Fig. 7*a*) be two sectors, and the angle ACB = TCS, then the arc A*m*B = arc T*v*S. For superimpose the sector ACB on TCS, so that AC may fall on TC, then CB must coincide with CS, because the angle ACB = angle TCS, AC = TC and BC = CS (*Def.* VIII. 1). Then the arc A*m*B will coincide with T*v*S, for if it fall inside on T*x*S, or outside on T*n*S, all the radii of the same circle cannot be equal, which is absurd (*Def.* VIII. 1); therefore the arc A*m*B = arc T*v*S.

2°. Let the arc A*m*B = arc T*v*S, and then the angle ACB = angle TCS. For, if not, let TCS be the greater, and let the angle *v*CS = angle ACB. Then, by the preceding, the arc *v*S = arc A*m*B = arc T*v*S; that is, a part is equal to the whole, which is absurd. Therefore the angle ACB = TCS, and the two sectors are equal in every respect.

It may be proved in the same way that sectors having equal angles in *equal* circles stand on equal arcs and reciprocally.

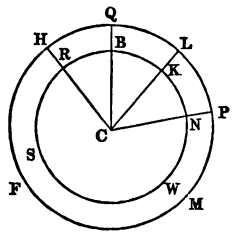
THEOREM.

6. *If radii, which divide any circle into equal sectors, meet the circumference of any concentric circle, they will divide the concentric circle into equal sectors, and its circumference into equal arcs.*

Let the radii CR, CB, CK, CN, &c., divide the circle SBW (Fig. 8) into equal sectors, and the same radii pro-

duced will divide the circle FQM into equal sectors, and consequently, its circumference into equal arcs. For, let sector QCL be superimposed on QCH and they will coincide. CQ and CL will respectively coincide with CQ and CH, because all are radii of the same circle, and the angle QCL = angle QCH. Also the arc QL must coincide with the arc QH. For if any part of the arc QL fall inside or outside of the arc QH, some of the radii drawn to the arc QL will be shorter or longer than the radii drawn to arc QH, which, since the point C remains fixed, is absurd (*Def.* VIII. 1). Hence the sector QCL = sector QCH, and the arc QL = arc QH. The same may be proved with regard to any other two sectors, as QCH and LCP, and their arcs. Hence, radii dividing any circle into equal sectors, divide every concentric circle, whose circumference they meet, into equal sectors, and its circumference into two equal arcs.

Fig 8.



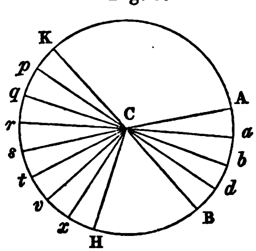
THEOREM.

7. *Angles at the centre of the same or equal circles are related to each other as the arcs contained between the radii which form the angles.*

If ACB and KCH (Fig. 9) be two central angles, then ACB : KCH : : arc AB : arc KH.

Let the arc AB be conceived as divided into m equal parts, Aa, ab, &c., and let the arc KH contain one of these parts n times, so that Aa = Kp = pq . . . . Draw to the points of division the radii Cp, Cq, . . Ca, Cb . . . Then the angle ACa = aCb = KCp = pCq . . . (5), and the angle ACB contains as many angles each equal to ACa, as the arc AB contains arcs each equal to Aa. In like manner, since the arc KH con-

Fig. 9.



tains n arcs each equal to Kp, the angle KCH contains n angles each equal to KCp. Now,

$$\text{arc AB} = \text{arc A}a \times m,$$

and

$$\text{arc KH} = \text{arc A}a \times n;$$

therefore

$$\frac{\text{AB}}{\text{KH}} = \frac{\text{A}a \times m}{\text{A}a \times n} = \frac{m}{n}. \quad \ldots (1)$$

Also

$$\text{angle ACB} = \text{angle AC}a \times m,$$

and

$$\text{angle KCH} = \text{angle AC}a \times n;$$

therefore,

$$\frac{\text{ACB}}{\text{KCH}} = \frac{\text{AC}a \times m}{\text{AC}a \times n} = \frac{m}{n}. \quad \ldots (2)$$

Hence from (1) and (2) $\frac{\text{ACB}}{\text{KCH}} = \frac{\text{AB}}{\text{KH}}$ or

$$\text{angle ACB} : \text{angle KCH} :: \text{arc AB} : \text{arc KH}.$$

It is supposed in the foregoing demonstration that it is possible to find an arc Aa which is contained a certain number of times exactly in each of the arcs AB and KH; that is, AB and KH are supposed to be *commensurable* arcs. But it may happen that the proposed arcs are *incommensurable*, or such as have no common divisor, however small. In this case also the corresponding central angles, which are likewise incommensurable, have the same ratio as the arcs.

Let AB and KH (Fig. 10) be two incommensurable arcs, then

$$\text{ACB} : \text{KCH} :: \text{AB} : \text{KH}.$$

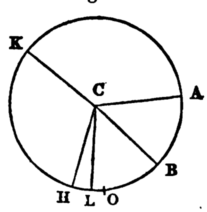
For, if not, let

$$\text{ACB} : \text{KCH} :: \text{AB} : \text{KO} > \text{KH},$$

or

$$\frac{\text{ACB}}{\text{KCH}} = \frac{\text{AB}}{\text{KO}} \quad \ldots (1)$$

Fig. 10.



Conceive AB divided into a number of equal parts, each part being less than HO, and let one of these be applied to KO (3), so as to mark off equal arcs, commencing with K; then one, at least, of the

points of division must fall between H and O. Let this point be L. Join C and L, and since AB and KL are commensurable arcs, by the preceding

$$ACB : KCL :: AB : KL,$$

or

$$\frac{ACB}{KCL} = \frac{AB}{KL}. \qquad \ldots (2)$$

But the equations (1) and (2) are clearly incompatible, for since the numerators on the corresponding sides of the equations are the same, if the denominator KCH < KCL, the denominator KO must be less than KL. Hence the hypothesis KO > KH is absurd; and in a similar way it may be proved that the fourth term of the assumed proportion cannot be less than KH; therefore it must be equal to it. Consequently,

$$ACB : KCH :: AB : KH.$$

8. Since, by the last theorem, KCH : ACa :: KH : Aa (Fig. 9), if the angle ACa be taken as the *unit* of angular measurement, then KCH = unit angle $\times \dfrac{KH}{Aa}$; that is, the angle KCH is equal to the unit angle taken as often as the arc KH contains the arc Aa. In other words, the same abstract number that represents how often KH contains Aa, also represents how often the angle KCH contains the unit angle; and it is in this sense that the arc KH is called the *measure* of the angle KCH.

The angle adopted by geometers from the earliest times, as the unit of angular measurement, is one whose sides intercept the 360th part of the circumference of a circle described with the vertex of the angle as centre. Thus if Aa be the 360th part of the entire circumference, the angle ACa is the *unit angle.* The 360 equal parts into which the circumference of every circle is supposed to be divided are called *degrees;* each degree is divided into 60 equal parts called *minutes;* and each minute into 60 equal parts called *seconds.* Symbols are used to express degrees, minutes, and seconds. Thus $57° \, 17' \, 45''$ means 57 degrees, 17 minutes, 45 seconds. When smaller divisions than seconds are to be expressed, decimal parts of a second are employed.

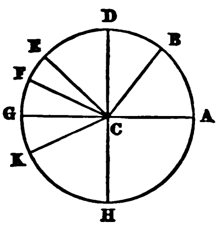
The number 360, though its selection was perfectly arbitrary, has the convenience that it admits of division without

remainder by all the integral numbers from 1 to 12 inclusive, except 7 and 11; which cannot be said of any smaller number.

I. The arcs contained between two radii are of the same number of degrees on every concentric circle (6); hence the measure of a central angle may be taken on any of the concentric circles.

II. An angle is said to contain as many degrees, minutes, and seconds as are contained in the arc which measures it. An angle of 90° is called a *right* angle. An angle of less than 90° is called an *acute* angle. An angle of more than 90° is called an *obtuse* angle. Thus if DBA be an arc of 90°, GA being a diameter of the circle (Fig. 11), the angle DCA is a *right* angle; BCA is an *acute* angle; and ECA is an *obtuse* angle.

Fig. 11.



III. What an acute angle or arc wants of 90° is called its *complement;* and what an angle or arc wants of 180° is called its *supplement.* A right angle has no complement, and is equal to its supplement.

IV. All right angles are equal, because they have equal measures.

Theorem.

9. *The sum of all the angles made on the same side of a straight line, by any number of straight lines meeting that line at the same point, is equal to* 180°, *or two right angles.*

Let FC, EC, DC, &c., be drawn to the point C of the line GA (Fig. 11), and on the same side of it; also, let a circle be described with C as centre, and any radius CA. The sum of the angles GCF, FCE, ECD, &c., contains the same number of degrees as the sum of the arcs which measure these angles respectively (8 II.); but the sum of the arcs GF, FE, ED, &c., is the arc GDA = 180° (2), since GA is a diameter of the circle.

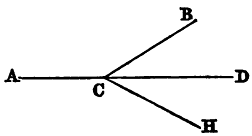
Cor. i. If lines KC, HC, &c., be drawn to the same point C on the lower side of the line GA, the sum of the angles GCK, KCH, &c., is also 180° ; hence, *the sum of all the angles made by any number of straight lines in the same plane, which meet at the same point, is equal to* 360°, *or four right angles.*

THEOREM.

10. *If a straight line meet two others at the same point, and on each side of it make an angle with one of these lines, and if the sum of the two angles be equal to* 180°, *the two straight lines shall form one and the same straight line.*

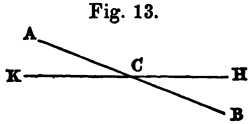
Let BC (Fig. 12) meet the two lines AC and HC, making BCA + BCH = 180°, then must HC coincide with CD, the production of the line AC. For by the preceding theorem (9), BCA + BCD = 180° ; but, by hypothesis, BCA + BCH = 180°, therefore, BCD = BCH, or CH must coincide with CD.

Fig. 12.



The angles BCA and BCD, which the line BC makes with the line AD, are called *adjacent* angles. When two straight lines, as AB and HK (Fig. 13), cut each other, the angles ACH and KCB are called *vertically opposite angles*, as are also the angles ACK and HCB.

Fig. 13.



THEOREM.

11. *When any straight line cuts another, the vertically opposite angles are equal.*

Let AB and HK cut each other in the point C, then ACH = KCB, and ACK = HCB. For ACH + HCB = 180° (9) ; and also ACH + ACK = 180° (9). Therefore, ACH + HCB = ACH + ACK ; and, subtracting ACH from each side of the equation, HCB = ACK. In the same way, HCB + ACH = HCB + KCB, therefore, ACH = KCB.

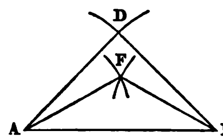
Problem.

12. *To find two points, each of which is equidistant from the extremities of a given line, or from any other two of its points.*

A line is said to be *given* when its position and length are known.

Let A and B (Fig. 14) be the extremities, or any two points of the given line. With A as centre, and any radius greater than one-half of AB, describe an arc of a circle (Postulate 3); also, with B as centre, and the same radius, describe another arc cutting the former at D. Then DB = DA (*Def.* VIII. 1). Again, with A as centre, and any other radius greater or less than AD, but greater than one-half AB, describe an arc of a circle; and with B as centre, and the same radius, describe another arc cutting the former at F. Then FA = FB (*Def.* VIII. 1). Hence D and F are the points required.

Fig. 14.



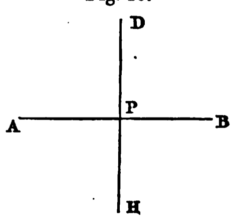
N. B.—In constructing diagrams in elementary geometry, the only instruments allowed, by most geometers, are the *ruler* and *compasses*—the ruler to guide the pencil in drawing straight lines, the compasses in describing circles. Both are virtually conceded in Postulates 1, 2, and 3.

PERPENDICULAR AND OBLIQUE LINES.

13. *Def.* A straight line is said to be *perpendicular* to another when it meets it with equal divergency on both sides; that is, when it makes the adjacent angles equal. A line is *oblique* to another when it meets it with unequal divergency.

I. Since (Fig. 15) APD + BPD = 2 right angles (9), if APD = BPD, each of these is a right angle. Hence, when a line is perpendicular to another, the angles which it makes with the latter are right angles. And if DPB be a right angle, APD must also be a right angle (9); that is, if a line meets another line

Fig. 15.



and makes a right angle with it, it is perpendicular to that other line (8, IV.)

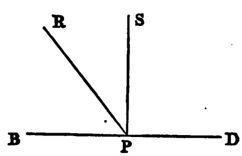
II. If DP be produced to H, then BPD = APD = BPH (11). Therefore, BP is perpendicular to DH, and the two lines DH and BA, are reciprocally perpendicular to each other.

The facility with which the eye can recognise a right angle, and the clear conception that the mind forms of its magnitude by considering the case of one line perpendicular on another, would suggest the convenience of adopting the right angle as the unit or standard whereby to estimate the magnitude of any other angle. An angle of 45°, for instance, is half a right angle; an angle of 30° is one-third of a right angle; an angle of 15° is one-sixth of a right angle, and so on. We shall retain, however, the unit previously mentioned (8), bearing in mind that any angle expressed in terms of that unit is readily expressed in terms of the right angle by dividing by 90.

Theorem.

14. *Only one perpendicular can be drawn to a line at a given point in it.*

Fig. 16.



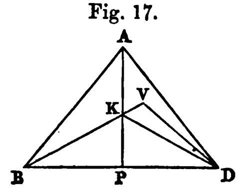
If possible, let SP and RP (Fig. 16) be both perpendicular to BD at the point P. Then SPB = a right angle = RPB (8, IV.), which is absurd.

N. B.—All the lines are supposed to be drawn in one plane; and the same is supposed in all the constructions as far as the section on *Planes*, unless where the contrary is expressly stated.

Theorem.

15. *A line perpendicular to another has every one of its points equidistant from any two points taken at equal distances from the meeting.*

Fig. 17.



Let AP (Fig. 17) be perpendicular to BD, and let B and D be at equal distances from P; then, every point in AP is equidistant from B and D. If there be any point in AP not equidistant from B and D, let it be the point A.

But AB = AD. For, let the figure APB be superimposed on APD, and AP and PB will respectively coincide with AP and PD. Because, AP = AP, PB = PD, and (13) APB = APD. Therefore, BA must coincide with DA, and be equal to it. Hence, there cannot be a point in AP which is not equidistant from B and D.

Theorem.

16. *A line bisecting another perpendicularly, and produced indefinitely, passes through all the points which are equidistant from the extremities of the other line.*

Let AP (Fig. 17) be perpendicular to BD, and bisect it, and there cannot be a point outside of AP which is equidistant from B and D. If there can be any, let it be the point V. Draw VB, DV, and DK. Then, BK + KV = BV = DV, and (15) BK = DK. Therefore, DK + KV = DV, which is absurd (*Axiom* 2).

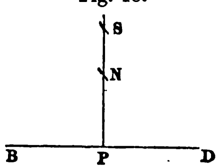
Cor. Hence, a straight line bisecting a chord perpendicularly must pass through the centre of the circle, because, the centre is equidistant from the extremities of the chord (*Def.* VIII. 1).

Theorem.

17. *A straight line which has any two of its points equidistant from the extremities of another line bisects it perpendicularly.*

Let SP (Fig. 18) have two points S and N equidistant from B and D, and SP bisects BD perpendicularly. If not, some other line may bisect BD perpendicularly. But no other line can bisect BD perpendicularly. For any line bisecting it perpendicularly (16) must pass through all the points equidistant from B and D, and therefore must pass through S and N, and (*Def.* V. 2) coincide with SP. Therefore, SP is the only line which can bisect BD perpendicularly.

Fig. 18.



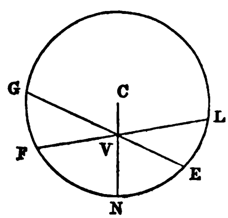
Cor. Hence, a radius bisecting a chord is perpendicular on it. Because the centre and the point at which the radius meets the chord are two points of the radius which are equidistant from the extremities of the chord.

THEOREM.

18. *Two chords in a circle which bisect each other are diameters.*

Let GE and FL (Fig. 19) be two chords bisecting each other at V. If they are not diameters, draw the radius CN through the point V. Then, by the last *cor.*, CVE = 90° = CVL, which is absurd. Therefore, two chords in a circle cannot bisect each other unless they meet in the centre and be diameters.

Fig. 19.



PROBLEM.

19. *To bisect a straight line perpendicularly.*

Find two points, each of which is equidistant from its extremities (12), draw a straight line through these two points, and it is the perpendicular required (17).

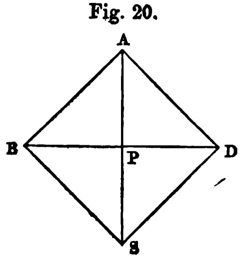
THEOREM.

20. *A line perpendicular to another, and having one of its points equidistant from the extremities of the other, bisects it.*

Let AP (Fig. 20) be perpendicular to BD, and let AB = AD, then BP = DP.

Produce AP, so that PS = AP. Draw BS and DS. Then BS = BA (15), and DS = DA. Therefore, BS = DS. Hence, A and S are two points in AS, each of which is equidistant from B and D; consequently (17) AS bisects BD perpendicularly.

Fig. 20.



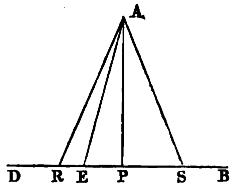
Cor. A radius perpendicular to a chord bisects it.

THEOREM.

21. *From the same point outside a given line only one perpendicular can be drawn to it.*

If more than one perpendicular can be drawn from a given point to the same straight line, let A be the point and DB (Fig. 21) the straight line. Let AP be one of the perpendiculars, and take two points, R and S, in the line DB, so that PR = PS, by describing a circle with P as centre (*Postulate* 3). Then AR = AS (15). If possible, let AE be also perpendicular to DB. Then (20) ER = ES. Therefore RS is bisected at P and E, which is absurd. Hence, only one perpendicular can be drawn from a given point to the same line.

Fig. 21.



Cor. Hence, two lines perpendicular to the same line can never meet, though produced indefinitely.

THEOREM.

22. *There cannot be more than two equal straight lines drawn from the same point to a given line.*

If it be possible to draw three equal straight lines from the same point to a given line, let A be the point, and BD (Fig. 21) the line, and let the three equal lines be AR, AE, and AS; then, a perpendicular may be drawn from A to the middle point between R and E (17), another from A to the middle point between R and S, and a third from A to the middle point between E and S. Therefore three perpendiculars may be drawn from A to BD, which is absurd (21).

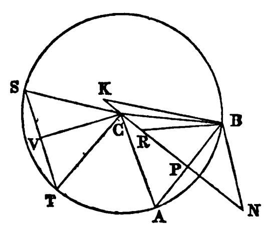
Cor. Hence, a straight line cannot cut the circumference of a circle in three points, for the three radii drawn to the three intersections would be three equal straight lines from the centre to the same straight line.

THEOREM.

23. *Equal chords in the same or equal circles subtend equal arcs.*

Let ST and AB (Fig. 22) be equal chords of the circle BAS, and then the arc AB is equal to the arc ST. Draw CV perpendicular to ST, and CP perpendicular to AB. Superimpose the figure CST on CAB, and ST will coincide with AB, CV will *fall on* CP, because (20. *Cor.*) SV = AP and SVC = APC (13. i.), then CT must coincide with CB; if not, let it coincide with BR or BK. Produce CP so that PN = CP, and draw BN; then if CT coincide with BR, BR = BC = BN (15); and if CT coincide with BK, BK = BC = BN (15); therefore if CT fall inside or outside of CB, there may be three equal straight lines from B to KN, which is absurd (22). Hence CT must coincide with CB, and consequently, CS must coincide with CA; therefore the angle SCT = ACB, and the arc ST = arc AB (7). It may be shown in the same manner that equal chords in *equal* circles subtend equal arcs.

Fig. 22.

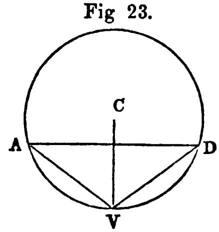


THEOREM.

24. *A radius perpendicular to a chord bisects the arc which the chord subtends.*

Let CV (Fig. 23) be a radius perpendicular to a chord AD, then the arc AV = arc DV; because the radius CV bisects the chord AD (20. *Cor.*), and therefore the chord AV = DV (15); consequently, by the last theorem, the arc AV = arc DV.

Fig 23.



THEOREM.

25. *The perpendicular is the shortest line that can be drawn from the same point to a given line.*

Let AP (Fig. 20) be perpendicular to BD, and it is the shortest line that can be drawn from A to BD; if not, some oblique line, as AB, may be as short as AP. Produce AP so that PS = AP, join B and S; then AB = BS (13, II.) (15), and 2AB = BS + AB > AS (*Ax.* 2). Therefore AB > $\frac{AS}{2}$. But $\frac{AS}{2}$ = AP, hence AB > AP, and consequently, AP is the shortest line from A to BD.

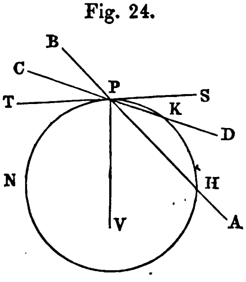
Cor. Since by the *distance* of a point from a line is meant the *shortest* length between them, hence *a perpendicular is the proper measure of the distance from a point to a line.*

THE SECANT, TANGENT, AND NORMAL.

26. *Def.* I. A straight line which cuts one or more lines either straight or curved is called a *secant.*

If P, one of the points in which the secant BA cuts the circle NPK (Fig. 24), be conceived as fixed, and the secant revolve round it, the other point H will move along the arc HKP and finally coincide with P. The secant will then take the position ST, and touch the circle only in a single point; for, during the revolution no point of the secant can touch the circle in the direction PN (22. *Cor.*). In this *ultimate* position the secant is called a *tangent.* Hence we get the following brief definition :—

Fig. 24.



Def. II. A *tangent* is a straight line which touches the circle in one point only.

This point is called the *point of contact.*

Def. III. A line perpendicular to the tangent and passing through the point of contact is called *a normal.*

Theorem.

27. *A radius to the point of contact is perpendicular to the tangent.*

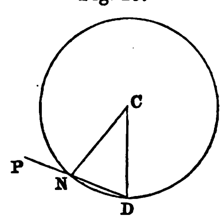
A radius to the point of contact is the *shortest* line that can be drawn from the centre to the tangent; for any other line from the centre should meet the tangent outside the circumference, therefore (25) the radius to the point of contact is perpendicular to the tangent.

Cor. The normal must pass through the centre; for, if not, a radius to the point of contact would be a second perpendicular to the tangent at the same point, which is absurd (14).

Theorem.

28. *A line perpendicular to a radius at its extremity is a tangent.*

Fig. 25.



Let DP (Fig. 25) be perpendicular to CD at the point D; if DP be not a tangent, let it cut or touch the circle at some other point N. Join C and N, and CN = CD; therefore CD is not the *shortest* line from C to DP, which is absurd (25). Hence

Cor. Only one tangent can be drawn to a circle at the same point (14).

But the same straight line may be a tangent common to an indefinite number of circles.

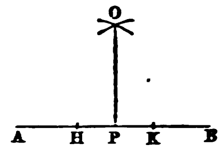
Problems.

29. I. *To draw a perpendicular to a line at a given point in it.*

Let AB (Fig. 26) be the line to which the perpendicular is to be drawn, and P the given point in it.

Construction.—Open the compasses, and, placing one of the legs on the point P, describe a circle cutting AB in the points H and K. Then, opening the legs of the compasses still wider, place one of them on the point H and describe an arc; transfer the leg of the compasses from H to K, leaving the opening unaltered, and describe another arc cutting the former in O. Draw a straight line through O and P, using the ruler to guide the pencil, and OP is the perpendicular required.

Fig. 26.



Proof.—By the construction PH = PK ; also, O is equidistant from H and K, since the arcs of the circles of which H and K are the centres were described with the same radius. Hence the line OP has two points each of which is equidistant from H and K; it is, therefore, perpendicular to AB (17).

The foregoing construction will serve as a type of the method to be adopted in the solution of other problems.

II. *From a given point outside a line to draw a perpendicular to the line.*

From that point as centre describe a circle cutting the line in two points, and find another point equidistant from the two intersections (12). The straight line joining this and the given point is the perpendicular required (17).

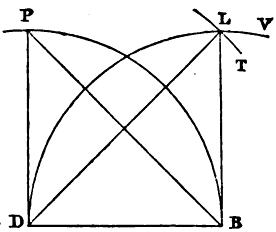
If the point from which the perpendicular is to be drawn be not determined, any point outside the line may be assumed, and the solution is then the same as before.

III. *To draw a straight line to the extremity of another, which cannot be produced, so that the former shall be perpendicular to the latter.*

Let DB (Fig. 27) be a straight line which cannot be produced, and let it be required to draw another line to B, which will be perpendicular to DB. With B as centre,

and any radius DB, describe an arc DV of a circle; and with D as centre, and DB as radius, describe another arc BP. Erect a perpendicular on DB at D (29. I.). Draw PB, and with D as centre, and PB as radius, describe an arc LT. Join B and L, and BL is perpendicular to DB at B. For, the chord DL = BP (*Def.* VIII. 1). Therefore, the arc BP = DL (23), and the angle BDP = DBL = 90°. Hence (13. I.), BL is perpendicular on DB.

Fig. 27.



IV. *To draw a tangent to a circle at a given point.*

Draw a radius to the given point, erect a perpendicular on it at its extremity (by the last), and it is the tangent required (28).

V. *To find the centre of a given circle.*

Draw two chords from any point in the circumference, bisect these chords perpendicularly (19), and the point at which the perpendiculars must cut each other is the centre of the circle (16. *Cor.*)

PARALLEL LINES.

30. *Def.* Two straight lines which are in the same plane and do not meet, though produced indefinitely both ways, are said to be *parallel.*

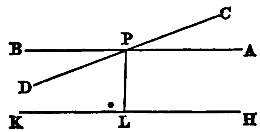
I. When two lines in the same plane are both perpendicular to a third line, they can never meet (21. *Cor.*); hence such lines are parallel.

II. We shall assume as evident the truth of the following statement:—

Axiom. 3. A line which meets one of two parallels if sufficiently produced shall meet the other.

From this it follows that through the same point only one line can pass so as to be parallel to a given line; for, if HK and AB (Fig. 29) be parallel, CD, which meets the latter, must, by the axiom, meet HK if both be sufficiently produced; it cannot, therefore, be parallel to HK and cut AB in P.

Fig. 29.



III. Two lines which are respectively parallel to a third are parallel to each other; for if the lines can meet when produced on either side, then through the same point two lines can pass both parallel to the third line, which, by the last, is impossible.

Theorem.

31. *A line perpendicular to one of two parallels is perpendicular to the other.*

Let AB and HK (Fig. 29) be parallel, and let PL be perpendicular to HK, it is also perpendicular to AB. If not, draw CD perpendicular to PL (29, I.), then CD and HK are parallel (30, I.); but, unless CD coincide with AB, if produced it will cut HK (30, II., *Ax.*); consequently CD must coincide with AB, which is, therefore, perpendicular to PL. And since two lines are reciprocally perpendicular, (13, II.), PL is perpendicular to AB.

Theorem.

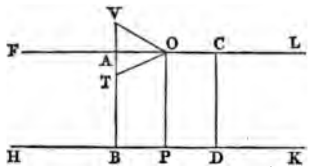
32. *The parts of perpendiculars intercepted between the same parallels are equal.*

Let AB and CD (Fig. 30), be two perpendiculars between the parallels FL and HK, then AB = CD.

Bisect BD in P perpendicularly (19), then OP the bisecting line is also perpendicular to AC (31). Let the figure OPDC be conceived to revolve round the line OP until it fall

on the figure OPBA; then, since OPD = OPB, the line PD will fall on PB, and, being equal to it, the point D will fall on B; also, because the angle PDC = PBA, each being a right angle, the line DC will fall on BA. Now if DC be not equal to BA, let it be equal to BT, or BV; in the first case OC must coincide with OT, in the second with OV; and since OCD is a right angle, OTB or OVB must be a right angle; but OAB is also a right angle; therefore, unless OC fall on OA, through the point O, two lines (OA and OT, or OA and OV) can pass, each of which is perpendicular to AB, which is impossible (21). Hence OC must fall on OA and therefore DC = BA.

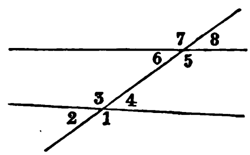
Fig. 30.



Cor. Hence two parallel lines are at the same distance from each other at all their points; for the distance of one line from another at any point is determined by letting fall a perpendicular from that point to the other line.

33. *Def.* When a secant cuts two other straight lines, the angles which it makes with them have received special names. They are called *alternate interior* when they are between the two lines and on opposite sides of the secant, as (Fig. 31) 3 and 5, 4 and 6; *alternate exterior* when they are outside the two lines and on opposite sides of the secant, as 1 and 7, 2 and 8; and *corresponding* angles when they are on the same side of the two lines and on the same side of the secant, as 1 and 5, 2 and 6, 3 and 7, 4 and 8.

Fig. 31.



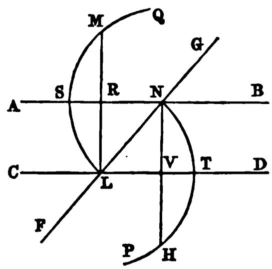
Theorem.

34. *A secant makes, with two parallels, equal alternate interior, equal alternate exterior, and equal corresponding angles.*

Let AB and CD (Fig. 32) be two parallels cut by GF, and then, NLT = LNS, NLC = LNB, GNB = CLF, GNS

= DLF, GLD = GNB, &c. Draw NV and LR perpendiculars between the parallels, and (32) NV = LR. With N as centre, and NL as radius, describe an arc LQ of a circle, and with L as centre, and LN as radius, describe another arc NP. Produce LR to meet the arc LQ at M, and NV to meet the arc NP at H. Then NH = 2NV = 2LR = LM (20 *cor.*) Therefore, the arc NT = arc $\frac{NH}{2}$ = arc SL = arc $\frac{LM}{2}$ (24). Consequently, the angle NLT, measured by the arc NT (8) is equal to LNS, measured by arc SL = arc NT. Also, LNS + LNB = 180° = NLT + NLC (9), therefore LNB = NLC.

Fig. 32.



Again, LNS = GNB, and NLT = CLF (11), therefore GNB = CLF; and since GNB + GNS = CLF + FLT (9), therefore GNS = FLT.

Finally, GNB = LNS = NLT, CLF = NLT = LNS, &c.

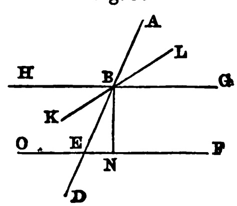
Cor. From the demonstration it is evident, that if a secant make the alternate interior angles equal, it must make the alternate exterior angles equal, and also the corresponding angles; and reciprocally.

THEOREM.

35. *If a straight line cuts two other straight lines and makes with them equal alternate or equal corresponding angles, these two lines are parallel to each other.*

Let AD cut HG and OF (Fig. 33), and make HBE = BEF; then HG must be parallel to OF. Draw BN perpendicular to OF, and if HG be not parallel to OF, draw (29, III.) KL perpendicular to BN, and (30, I.) KL is parallel to OF; therefore KBE = BEN = HBE, that is, a part of HBE is equal to it, which is

Fig. 33.



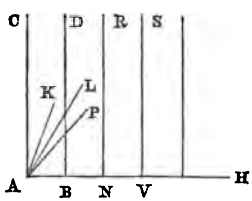
absurd. Therefore, no line but HG can be perpendicular to BN, at B, or parallel to OF.

Since the equality of the alternate exterior or corresponding angles involves also the equality of the alternate interior (34 *Cor.*), hence, when a secant makes equal alternate exterior or equal corresponding angles with two lines, these lines are parallel.

Since OEB + BEF = two right angles (9), and BEF = HBE (Fig. 33) when GH and FO are parallel; therefore HBE + OEB = two right angles. Hence, if KBE + OEB be less than two right angles, the lines LK and FO, if sufficiently produced, must meet on that side of the secant on which the sum of the two interior angles is less than two right angles. This is Euclid's famous axiom, if indeed it can be called an axiom, and on it rests his whole theory of parallels. The assumption in the text (30 *Ax.* 3), though much less objectionable, is also, it must be admitted, somewhat wanting in the amount of self-evidence required for an axiom in so strict a science. The ablest Geometers have endeavoured to dispense with these axioms, but their attempts have ended either in other assumptions fully as great as these, or proofs have been given of them so lengthy and complicated, as to be quite unsuited for elementary teaching. The following proof of *Axiom* 3 (30), though it involves ideas which the beginner will find a little difficult to realise, is, perhaps, one of the simplest that can be given of it.

Let AC and BD be two parallel lines, and let AH be drawn at right angles to them (31). Conceive the right angle CAH as divided into n equal parts by the lines AK, AL, &c.; and draw NR, VS, &c., perpendicular to AH, so that AB = BN = NV, &c.; also, let the number of these equal parts be n. Produce indefinitely all the lines in the directions AC, BD, . . AH, AK, AL, &c., respectively. Then the indefinite space contained by the indefinite lines AC and AH is equal to n times the indefinite space contained by the lines AC and AK. Now the indefinite spaces contained by the lines AC and BD, BD and NR, NR and VS, &c., are all equal, since, if any one be properly applied to any other, they must coincide throughout their entire extent, the angles formed at the line AH being right angles by construction; therefore, the sum of these indefinite spaces is equal to n times the indefinite space contained by the lines AC and BD. But this sum is less than the indefinite space contained by the lines AC and AH; for *no finite number* of the indefinite spaces contained by the lines AC, BD, NR, &c., can fill up entirely the indefinite angular space CAH. Therefore, n times the indefinite space contained by the lines AC and AK is greater than n times the indefinite space contained by the lines AC and BD; consequently, the indefinite space contained by AC and AK is greater than the indefinite space contained by AC and BD; but this is obviously impossible, unless AK, when *produced*, cut BD which is parallel to AC.

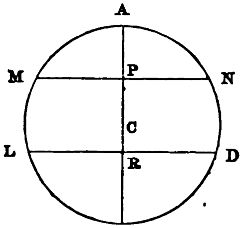
Fig. 35.



THEOREM.

36. *Two arcs of the same circle contained between two parallel chords are equal.*

Fig. 34.



Let MN and LD (Fig. 34) be two parallel chords, then arc ML = arc ND. Draw a diameter AB perpendicular to MN (29, II.); it is also perpendicular to LD (31). Now the arc BD = arc BL, and arc AN = arc AM (24). Therefore, 180° − BD − AN = ND = 180° −BL − AM = ML (2).

PROBLEMS.

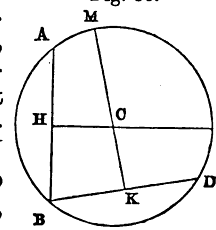
37. I. *To draw a line parallel to another through a given point.*

Erect a perpendicular on the given line at any assumed point in it (29, I.); from the *given* point draw a line at right angles to that perpendicular (29, II.), and it will be parallel required (30, 1).

II. *To describe the circumference of a circle through three given points which are not in the same straight line.*

Let A, B, and D bethe three given points. Join any one of them as B with the other two by straight lines; bisect these lines perpendicularly (19), and the bisecting lines, NH, MK, must meet. For, if not, they are parallel, and DB, which meets MK at right angles, if produced, will meet HN also at right angles (30, II., 31). But BA is perpendicular to HN, therefore, if KM and HN be parallel, through the point B, two lines can pass, both perpendicular to HN, which is impossible (21). Consequently, HN and KM must meet at some point C, equidistant from

Fig. 36.



A, B, and D (15). If, therefore, with C as centre, and the distance of any of the given points as radius, a circumference be described, it will pass through the three points A, B, and D.

Cor. i. Since the two perpendiculars KM and HN cannot intersect in more than one point (*Def.* V. 2), hence only one circle can be drawn through three given points.

Cor. ii. *Two circles cannot cut each other in more than two points;* because, if they can, two circles can be drawn through three given points.

ANGLES WHICH ARE NOT CENTRAL.

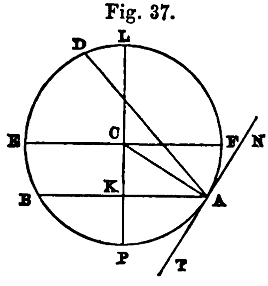
38. Besides central angles, formed by lines meeting each other at the centre of the circle, there are others which take different names according to the situation of their vertex with regard to the centre. An *angle of a segment* is that which is made by a tangent and a chord meeting the tangent at the point of contact; an *inscribed angle* is that which is formed by two chords meeting each other in the circumference; an *eccentric angle* is formed by two lines meeting each other between the centre and circumference; and a *circumscribed angle* is that which is made by two chords produced to meet each other outside the circumference.

THEOREM.

39. *The measure of an* angle of a segment *is half the arc which the chord subtends.*

The chord AB (Fig. 37) makes, with the tangent TN, two angles of a segment; one, BAT, measured by half the arc AB, the other, BAN, measured by half the arc ALB. Draw the radius CA to the point of contact, and the two diameters, FE parallel, and LP perpendicular, to the chord AB; then CAT = FCP, two right angles (27), and CAK

Fig. 37.



= FCA, two alternate interior angles (34). Hence the angle of a segment BAT = CAT − CAK = FCP − FCA = ACP. But the measure of the central angle ACP is arc AP = $\frac{1}{2}$AB (24). Therefore the angle of a segment BAT, which is equal to the central angle ACP, has also for its measure $\frac{1}{2}$AB.

The other angle of a segment BAN, and the central angle ACL, have, then, equal supplements, BAT and ACP, and, being equal (8, III.), are measured by the arc AL = $\frac{1}{2}$ALB. Besides, BAN = CAN + CAB = LCF + FCA = LCA measured by AL.

THEOREM.

40. *The measure of an* inscribed *angle is half the arc contained by its sides.*

The inscribed angle BAD (Fig. 37) is equal to the angle of a segment DAT, less the angle of a segment BAT; and, of course, its measure is the half of the arc DBA less the half of the arc BA (39), which is the half of the arc BD.

Cor. i. An angle of a segment and an inscribed angle are equal, if the arc subtended by the chord of the one is equal to the arc contained by the sides of the other.

ii. A central angle is double of an inscribed angle when its sides contain the same or an equal arc (8, 40).

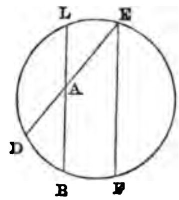
iii. An inscribed angle is right, acute, or obtuse, according as the arc contained by the sides is equal to, less or greater than the half of the circumference.

iv. Two chords from the extremities of the same diameter, and meeting at any point of the circumference, make an inscribed right angle, and are perpendicular to each other.

THEOREM.

41. *The measure of an* eccentric *angle is half the sum of the two arcs contained by the sides and by their prolongation.*

Fig. 38.



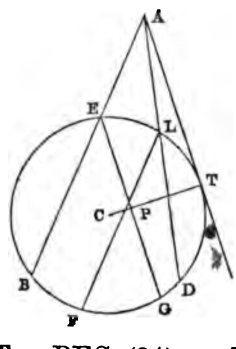
Produce to L and E (Fig. 38) the two sides of the eccentric angle BAD; and from either L or E, draw a chord EF parallel to one of the sides AB (37, I.). Then the eccentric angle BAD is equal to its corresponding inscribed angle FED (34), and its measure is $\frac{1}{2}$DF (40) = $\frac{1}{2}$DB + $\frac{1}{2}$BF = $\frac{1}{2}$DB + $\frac{1}{2}$EL (36).

THEOREM.

42. *The measure of a* circumscribed *angle is half the difference of the two concave and convex arcs contained by its sides.*

Let BAD (Fig. 39) be a circumscribed angle. From the intersection of either of the two secants with the circumference, draw the chord LF parallel to the other secant AB (37). Then the circumscribed angle BAD is equal to its corresponding inscribed angle FLD (34), and has the same measure, which is $\frac{1}{2}$FD = $\frac{1}{2}$BD − $\frac{1}{2}$BF = $\frac{1}{2}$BD − $\frac{1}{2}$EL (36).

Fig. 39.



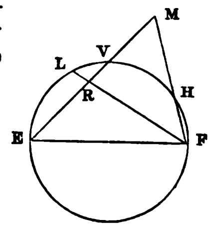
Cor. i. The circumscribed angle BAT, made by a secant and tangent, is also measured by half the difference of the concave and convex arcs.

Draw the chord EG parallel to the tangent, and the radius CT to the point of contact, which bisects the arc EG at T (24); then, the angle BAT = BEG (34), and has the same measure $\frac{1}{2}$BG = $\frac{1}{2}$BT − $\frac{1}{2}$GT = $\frac{1}{2}$BT − $\frac{1}{2}$ET (24).

ii. Any two lines perpendicular to each other, which pass by the opposite extremities of a diameter, must meet in the circumference of the circle.

Let two lines, perpendicular to each other, pass by the extremities E and F, of the diameter EF (Fig. 40), and these two lines must meet in the circumference of the circle FLE. If not, let them meet outside at M, or inside the circumference at R. If they meet at M, the angle $M = \frac{180°}{2} - \frac{VH}{2}$ (42); therefore $M < 90°$. If they meet at R, the angle FRE $= \frac{180°}{2} + \frac{LV}{2}$ (41), and therefore FRE $> 90°$. Hence neither of the angles can be of 90°, and consequently the lines drawn from F and E cannot be perpendicular to each other (13, 1), when they meet inside or outside the circumference.

Fig. 40.



Problems.

43. I. *To bisect a given angle or arc.*

From the vertex of the given angle as a centre describe a circle cutting the two sides; join the two intersections by a chord, and a line from the vertex, perpendicular to the chord, will bisect the chord, the arc it subtends, and the given angle measured by the arc (20, 24, 8).

When the arc is given, a line bisecting perpendicularly the chord which joins its extremities must pass through the centre and bisect it (16. *Cor.* 24).

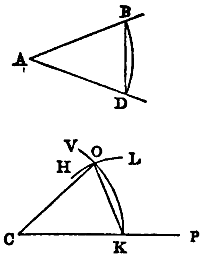
II. *To double a given angle.*

From the vertex as a centre describe a circle intersecting the two sides; from either of the intersections draw a chord perpendicular to the other side, and from the centre draw a radius to the extremity of the perpendicular chord. The radii at the extremities of the perpendicular chord form the double angle (24).

III. *To make at any point of a given line an angle equal to another given angle.*

Let BAD (Fig. 41) be the given angle and C the point in the line CP at which it is required to make an angle equal to the given angle BAD. With A as centre and any radius describe a circle cutting the sides of the angle in B and D, and draw the chord BD. Again with C as centre and the same radius as before describe the arc KV; also with K as centre and a radius equal to the chord BD describe the arc HL, cutting the arc KV in O, and draw the chord KO. Then since the chords KO and BD are equal chords in equal circles, the arcs KO and BD are equal (23), and therefore the central angles which these arcs respectively measure are equal (8), that is OCK = BAD.

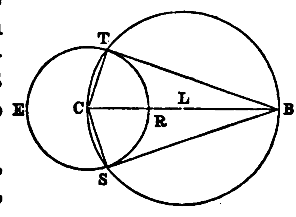
Fig. 41.



IV. *To draw a tangent to a circle from a given point outside the circle.*

Let TER (Fig. 42) be the circle and B the given point. Find the centre of the circle (29, V.), and from B draw BC to the centre C. Bisect BC in L (19), and with L as centre, and LC as radius, describe a circle CTB. Draw TB and SB, and they are tangents to the given circle.

Fig. 42.



Because since CTB = 90° (40, iv.), and CSB = 90° (40, iv.), BT and BS are respectively perpendicular to the radii CT and CS at their extremities, and consequently are tangents (28).

Cor. i. Only two tangents can be drawn from a given point to a given circle; because BT and BS are the only lines which can be drawn from B so as to be perpendicular

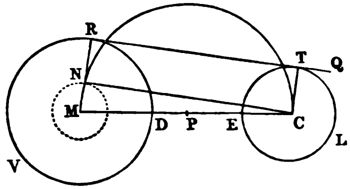
to radii of the circle TER at their extremities (42, ii.), since the two circles TER and TBS cannot cut each other in more than two points (37, II., *Cor.* ii.).

Cor. ii. Since CT = CS, therefore the arc CT = arc CS (23) ; consequently the angle CBT = CBS (40) ; that is, the two tangents make equal angles with the line which joins the centre of the circle and the external point from which the tangents are drawn.

V. *To draw a tangent common to two circles.*

Let RDV and TLE (Fig. 43) be the two circles. Join their centres M and C. Bisect CM in P, and with P as centre and CP as radius describe a circle. From M draw a chord MN = MD ∓ CE. Join C and N, and CNM = 90° (40, iv.). Produce, if necessary, MN to R. Draw RQ parallel to NC (37), and from C draw CT perpendicular to RQ. Then CT = NR (32) = CE. Hence as CT is a radius of one of the circles, and MR a radius of the other, the line RT which is perpendicular to MR and CT (13, II.) at their extremities, is a tangent common to the two circles (28).

Fig. 43.

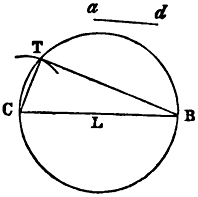


When MN = MD + CE, RQ will obviously touch TLE on the lower side.

VI. *From two given points to draw two lines perpendicular to each other, one of which shall be of a given length.*

Let C and B (Fig. 44) be the two given points, and *ad* the line of given length. Join C and B, and bisect BC in L. With L as centre and CL as radius describe a circle CTB; also with C as centre and *ad* as radius describe an arc cutting CTB in T. Draw CT and BT, and they are the perpendiculars required (40, iv.).

Fig. 44.



Exercises.

1. A straight line which joins any two points taken in the circumference of a circle must fall within the circle.

2. In a given circle to draw a chord equal to a given straight line which is less than the diameter of the circle.

3. Two circles which cut or touch each other cannot have a common centre.

4. From any point within or without a circle which is not the centre, two, and only two, equal straight lines can be drawn to the circumference.

5. When a secant meets two straight lines, if the sum of the two interior angles on the same side of the secant is equal to two right angles, the lines are parallel.

6. To draw a line from a given point outside a given line which will make a given angle with the given line.

7. If two circles touch, internally or externally, but do not cut each other, the straight line which joins their centres must pass through the point of contact.

8. The lines which bisect two adjacent supplementary angles are at right angles to each other.

9. A segment of a circle being given, to describe the circle of which it is a segment.

10. When two circles intersect, the arcs on the same side of their common chord cannot contain an equal number of degrees.

11. If a chord is drawn parallel to a tangent, the arcs intercepted between the point of contact and the extremities of the chord are equal.

12. When two circles intersect, the line which joins their centres bisects their common chord perpendicularly.

13. A diameter bisecting a chord which does not pass through the centre bisects all the chords parallel to it.

14. A tangent to the inner of two concentric circles is a chord of the outer one, prove that it is bisected at the point of contact.

15. If the arcs between two chords are equal, the chords are parallel.

16. The longest and shortest distances from a given point to the circumference of a circle are on the line which passes through the centre, whether the point be within or without the circle.

17. Prove that when one circle is entirely outside another, four common tangents can be drawn to the two circles.

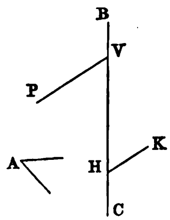
18. If from one of the points of intersection of two circles diameters be drawn, the straight line which joins the other extremities of the diameters shall pass through the other point of intersection.

19. When two angles have their sides respectively parallel, the angles are equal, or supplementary.

20. If two parallel lines are tangents to a circle, the points of contact are at the extremities of the same diameter.

In solving problems which occur in the exercises, the student will find it most convenient, as a general rule, to commence by supposing the problem solved, drawing such lines as the solution of the problem requires. Then examining or *analysing* the diagram, he will see certain relations of the lines, or properties of the figure, which will suggest a new method of construction satisfying the conditions of the problem. For instance, the problem in exercise 6 would be solved according to this method as follows:—

Fig. 44*a*.



Let P be a given point (Fig. 44*a*), BC a given line, and A a given angle, it is required to draw through P a line meeting BC, and making with it an angle equal to A. Let PV be such a line, then PVC = A.

Now it at once occurs to a mind familiar with the theorems previously established, that if an angle VHK equal to A were made at any point H along the line BC, one side of this angle should be parallel to PV (35). This suggests the following construction in the solution. At any point H in the given line make an angle BHK equal to the given angle A, through P draw PV parallel to HK, the side of the angle which does not lie in the given line, and PV is the line required. For (34) PVC = VHK = A.

SECTION II.

ON RECTILINEAR FIGURES.

44. Since two straight lines cannot meet each other in more than one point (*Ax.* 1), three, at least, are necessary to enclose a space, and form the simplest of all rectilinear figures called a *triangle*. A figure formed by four straight lines is called a *quadrilateral;* by five, a *pentagon;* by six, a *hexagon;* by seven, a *heptagon;* by eight, an *octagon;* and so on, their general name being *polygon*. The straight lines are the *sides*, and the sum of the sides is the *perimeter*, of the polygon.

THE TRIANGLE.

45. *Def.* A triangle, as its name implies, is a figure of three angles made by the meeting of three straight lines called its sides.

When the three sides of a triangle are equal, it is called *equilateral;* when only two of the sides are equal it is called *isosceles;* and when the three sides are unequal it is called a *scalene* triangle. On account of its angles, a triangle is called *right-angled, obtuse-angled,* or *acute-angled,* according as it contains a right angle, an obtuse angle, or has its three angles acute.

For distinction sake, one of the sides of a triangle is frequently called the *base*, and the point in which the other two sides meet each other is called the *vertex.*

In a right-angled triangle the side opposite the right angle is called the *hypothenuse.*

Since a straight line is the shortest that can be drawn between two points (*Axiom* 2), it follows that any one side of a triangle is necessarily less than the sum of the other two.

THEOREM.

46. *The sum of the three angles of every triangle is equal to the sum of two right angles, or* 180°.

Every triangle may have a circle described about it, the circumference passing through the vertex of each of the angles, since the vertices of the angles are three points not in the same straight line (37. II,). The angles are then inscribed angles, and the sides contain the entire circumference. Therefore, since each angle is measured by half the arc contained by its sides (40), the sum of the measures of the three angles is half the circumference, or 180°, the same as the sum of the measures of two right angles.

Cor. i. A triangle cannot have more than one right angle, or one obtuse angle.

Cor. ii. Any angle of a triangle is supplement to the sum of the other two, and reciprocally.

Cor. iii. Two triangles which have two angles respectively equal, or the sum of two of their angles equal, have the third angle equal.

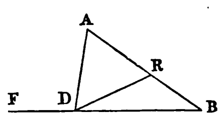
Cor. iv. In a right-angled triangle either of the acute angles is the complement of the other.

Cor. v. When a circle is described about a right-angled triangle, the hypothenuse is the diameter of the circle; for, since the three angles are inscribed angles, and the right angle is equal to the sum of the other two, the arc which the hypothenuse subtends must be equal to the sum of the arcs subtended by the other two sides; therefore, the hypothenuse bisects the circumference, and consequently is a diameter (2).

THEOREM.

47. *If any of the three sides of a triangle be produced, the exterior angle shall be equal to the sum of the two interior opposite angles.*

Fig. 45.



The exterior angle ADF (Fig. 45) + ADB = 180° (9), and ADB + A + B = 180° (46). Therefore, ADF = A + B.

Cor. The three sides of a triangle being produced, the sum of the three exterior angles is 360°.

The sum of every exterior and contiguous interior angle is 180°. Therefore the sum of the three exterior angles, made by the sides of the triangle produced, and of the three interior angles, is 3 × 180°, from which subtracting the sum of the three interior angles, or 180°, there remain for the sum of the three exterior angles, 360°.

THEOREM.

48. *In every triangle equal sides are opposite to equal angles, greater sides to greater angles, and reciprocally.*

1°. Equal sides are opposite to equal angles. For if a circle be described about any triangle containing equal angles,

the sides opposite to equal angles will be chords of equal arcs; because they will be the chords of arcs, the halves of which measure the opposite inscribed equal angles (40), and therefore these sides will be equal.

2°. Greater sides are opposite to greater angles. Let (Fig. 45) ABD be a triangle in which BDA > A: then AB > BD. Take ADR = A; and, by the last, DR = AR: then AR + RB = DR + RB > DB (*Ax.* 2). Therefore AR + RB = AB > BD.

3°. Equal angles are opposite to equal sides. For if any triangle having equal sides be inscribed in a circle, the angles opposite the equal sides will be inscribed angles opposite to equal chords and arcs (23), and therefore will be equal to each other (40).

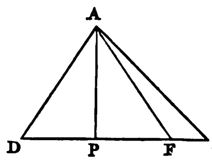
4°. Greater angles are opposite to greater sides. Let ABD (Fig. 45) be a triangle in which AB > AD, and then ADB > B. Take AR = AD; draw DR, and ADR = ARD = B + RDB (47). Therefore, ADB = ADR + RDB = B + RDB + RDB; that is, ADB is greater than B.

Cor. i. An equilateral triangle is also equiangular: an isosceles triangle has two equal angles opposite the two equal sides: a scalene triangle has its three angles unequal; and reciprocally.

THEOREM.

49. *If from any point in a perpendicular to a line oblique lines be drawn to the latter, the longest shall be that which intersects the line at the greatest distance from the perpendicular.*

Fig. 46.



1°. Let AP (Fig. 46) be perpendicular to BD, and AB and AF be oblique lines meeting BD *on the same side* of the perpendicular, then AB > AF.

Because the angle AFB = APB + FAP (47); therefore AFB > 90°; but ABF < 90° (46, *cor.* i.), consequently, AFB > ABF, and (48) AB > AF.

2°. Let the oblique lines be AB and AD *on the opposite*

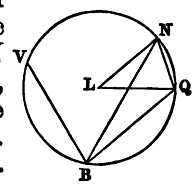
sides of the perpendicular; then, describing a circle with A as centre, and AD, one of the oblique lines, as radius, since AD is greater than AP (25), the circle must intersect DB, or DB produced, in some point, as F, making PF = PD, for AP when produced to the circumference becomes a radius, and DF a chord of the circle (20, *Cor.*) Therefore, as before, if PF < PB, AB > AF, and since AD = AF, consequently AB > AD.

THEOREM.

50. *Greater arcs of the same circle, or of equal circles, have greater chords, and reciprocally.*

1°. Let the arc BN (Fig. 47) be greater than arc BV, then the chord BN > BV. Describe a circle with B as centre and BV as radius cutting the circle NVB in the points V and Q; draw the chords BQ and NQ, and the radii LQ, LN. Then the arc BQ = arc BV (23), and the angle LQN = LNQ > BNQ. Therefore NQB > BNQ, and (48) the side BN > BQ. But, by the construction, BQ = BV, therefore BN > BV.

Fig. 47.



2°. Greater chords subtend greater arcs. If not, they must subtend arcs equal to, or smaller than, the arcs subtended by smaller chords. They cannot subtend arcs equal to those subtended by smaller chords (3, ii.); nor can they subtend arcs smaller than those of smaller chords, for then greater arcs would not have greater chords, which, by the first case, is impossible. Hence, greater chords subtend greater arcs.

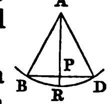
THEOREM.

51. *A line from the vertex of the unequal angle of an isosceles triangle, perpendicular on the opposite side, bisects both the angle and the side.*

If there be an isosceles triangle ABD (Fig. 48), in which AB = AD, and a circle be described with A as centre, and

AB as radius, the circumference will pass through B and D; also, the perpendicular AP, when produced to R, will be a radius, and the side BD, a chord of the circle. Therefore (20 *Cor.*) AR will bisect BD, and also the arc BRD (24), and consequently the central angle BAD measured by this arc.

Fig. 48.

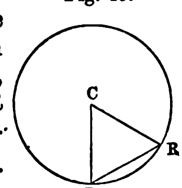


Cor. A perpendicular from any angle of an equilateral triangle to the opposite side bisects both the angle and the side. Because every one of the three angles may be taken for the vertex of an isosceles triangle.

THEOREM.

52. *The chord of an arc of* 60° *is equal to the radius of the circle.*

Fig. 49.



Let BR (Fig. 49) be the chord of an arc of 60°. Draw the radii CB and CR. Then CBR + CRB + BCR = 180° (46); that is, CBR + CRB + 60° = 180°. Therefore, CBR + CRB = 120°. But CBR = CRB (48), consequently 2 × CBR = 120°, or CBR = 60°. Therefore CR = BR (48).

EQUAL TRIANGLES.

53. Two triangles, or any two figures of the same number of sides, are *equal* when the sides and angles of one are equal to the sides and angles of the other, each to each; so that one being superimposed on the other, all the sides and angles coincide, two and two.

The sides opposite equal angles in different triangles are called *homologous*, or *corresponding* sides; and the angles opposite equal sides in different triangles are called *homologous* angles.

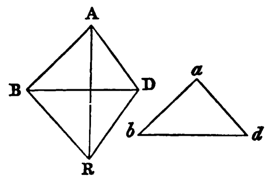
Two triangles are equal in the five following cases:—

THEOREM.

54. *Two triangles are equal which have an equal angle each contained by two sides respectively equal.*

Let BAD = *bad*, AB = *ab*, and AD = *ad* (Fig. 50). Apply the figure *bad* to BAD, so that *ab* may fall on AB, and *ab* being equal to AB will coincide with it. Also, because BAD = *bad*, the side *ad* will fall on, and coincide with, AD. Therefore (*Def.* V. 2) *bd* must coincide with BD, and the two triangles must be equal in every respect.

Fig 50.



THEOREM.

55. *Two triangles are equal which have an equal side each between two angles respectively equal.*

Let BD = *bd*, ABD = *b*, ADB = *d* (Fig. 50). The side *bd* being superimposed on BD will coincide with it; and, since ABD = *b*, and ADB = *d*, the sides *ba* and *da* must fall on BA and DA respectively. Therefore the point *a*, being in both the lines *ba* and *da*, must coincide with a point which is common to the lines BA and DA, that is, with A. Consequently the two triangles coincide, and are, therefore, equal.

THEOREM.

56. *Two triangles are equal which have two angles respectively equal, and an equal side each opposite either of the two angles.*

Let ABD = *b*, ADB = *d*, AB = *ab* (Fig. 50); then (46, *Cor.* iii.) A = *a*, and the case is the same as the last.

It must, however, be borne in mind, that unless the equal angles are similarly situated in reference to the equal sides, one of the triangles must be reversed before being superimposed on the other.

THEOREM.

57. *Two triangles are equal which have three sides equal, each to each.*

Let ABD and *abd* (Fig. 50) be two triangles in which AB = *ab*, BD = *bd*, AD = *ad*. Then the triangle ABD = *abd*. Let *abd* be applied to ABD, so that their longest sides, BD and *bd*, may coincide; let *ab* fall on BR, and *ad* on DR. Draw AR, then (48) the angle BAR = BRA, and the angle DAR = DRA. Therefore, DAR + BAR = DAB = DRA + BRA = BRD. Hence the triangles BAD and BRD have BAD = BRD, BA = BR, and AD = RD, and (54) are equal. Therefore the triangle *abd* = BRD = BAD.

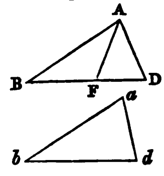
THEOREM.

58. *Two triangles are equal which have two sides respectively equal, and an equal angle each opposite either of the two sides, provided the angle opposite the other side is of the same kind in each.*

Two angles are said to be of the same kind when they are both equal to, greater, or less than 90°.

Let AB = *ab* (Fig. 51), AD = *ad*, B = *b*, and D of the same kind as *d*. Then the triangle ADB = triangle *adb*. Superimpose *ab* on AB, and being equal they will coincide; also, *bd* will fall on BD, because B = *b*. Then *ad* will coincide with AD, or with a line from A to BD equal to AD. Let AF be equal to AD, and no other line can be drawn from A to BD which will be equal to AD (22). Hence *ad* must coincide with AD, or with AF. But if D and *d* be angles of the same kind, *ad* cannot coincide with AF. Because BFA is the supplement of AFD (9), and AFD = D (48). Hence

Fig. 51.



BFA and D are angles of different kinds; consequently *d* cannot be equal to BFA, if D and *d* be of the same kind. Therefore *ad* cannot, in that case, coincide with AF, but must coincide with AD. Hence, if D and *d* are angles of the same kind, the triangle ABD = the triangle *abd*.

59. From the five preceding theorems it follows that two triangles are equal which have any *three* things equal of which one is a side; except when the two triangles have two sides respectively equal, and an equal angle each opposite to one of the sides. Then, to prevent ambiguity, a *fourth* datum is necessary; for the triangles are not equal unless the angle opposite the other side be of the same kind in both. If, however, the equal angles are right angles, or obtuse angles, or if the two triangles are acute-angled triangles, this last condition is obviously present (46, *Cor.* i.). Hence

Cor. i. *Two right-angled triangles are equal which have, besides the right angle, any two things equal, one of which is a side.*

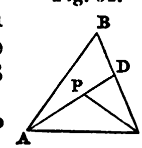
Cor. ii. In equal triangles the sides opposite to equal angles are equal; because equal triangles, if superimposed, coincide, and therefore the sides opposite to the equal angles coincide and are equal.

Cor. iii. Two triangles may have their three angles equal, two and two, without being equal. Because if from any point in the line *ab* (Fig. 51) a line be drawn parallel to *bd* and intersecting *ad*, a triangle will be formed which, though only a part of *abd*, will be equiangular with it (34).

THEOREM.

60. *If two lines be drawn to a point within a triangle from the extremities of one of the sides, the sum of these lines shall be less than the sum of the other two sides of the triangle, but the lines shall contain a greater angle.*

Fig. 52.



Let the lines AP and CP (Fig. 52) be drawn to the point P from the extremities of the side AC of the triangle ABC, then AP + PC < AB + BC, but APC > ABC.

1°. Produce AP to D, then AB + BD > AP

+ PD (*Ax.* 2) and PD + DC > CP. Therefore AB + BD + PD + DC > AP + PD + CP, or, subtracting PD from each side of the inequality and putting BD + DC = BC, AB + BC > AP + CP.

2°. APC > ADC (47), and ADC > ABC, therefore APC > ABC.

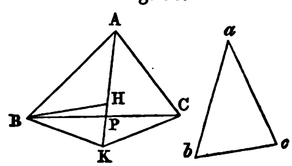
Theorem.

61. *If two triangles have two sides each respectively equal, but the angles which these sides contain unequal, the third side shall be greater in that triangle whose two sides contain the greater angle, and reciprocally.*

1°. Let ABC and *abc* (Fig. 53) be two triangles in which AB = *ab*, AC = *ac*, but BAC > *bac*, then BC > *bc*.

Apply the triangle *abc* to the triangle ABC so that *ab* may fall on AB, then *ab* will coincide with AB, and, since *bac* < BAC, the side *ac* will lie between the sides BA and AC, and coincide with AH, AP, or AK. Then *bc* will take the position BH, BP, or BK. But AH + BH < AC + BC (60), and AH = *ac* = AC, therefore BH < BC.

Fig. 53.



BP is also necessarily less than BC.

Thirdly, if *bc* take the position BK, then BKC > AKC; but (48) AKC = ACK > BCK; therefore BKC > BCK, and consequently BC > BK (48). Hence, since *bc* must coincide with BH, BP, or BK, *bc* is less than BC.

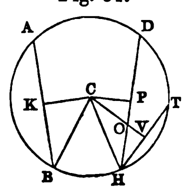
2°. If AB = *ab*, AC = *ac*, but BC > *bc*, then BAC > *bac*. For, if not, either BAC = *bac*, or BAC < *bac*; but, in the first case, BC = *bc* (54), and, in the second, *bc* > BC, by the first part of the present theorem. Therefore BAC cannot be equal to, nor less than *bac*, consequently it must be greater than it.

Theorem.

62. *Two chords equidistant from the centre are equal and reciprocally.*

1°. Let AB and HD (Fig. 54) be two chords equidistant from C, then AB = HD.

Fig. 54.



Draw from the centre CK and CP perpendicular to AB and HD respectively (29, II.); also the radii CB and CH. Then, since the chords are equidistant from the centre, CK = CP (25, *Cor.*). Therefore the triangles CKB and CPH are equal (59, *Cor.* i.), and BK = HP. Consequently $2 \times BK = 2 \times HP$; but $2 \times BK = AB$, and $2 \times HP = HD$ (20, *Cor.*), therefore AB = HD.

2°. If AB = HD, the two chords are equidistant from the centre; for, with the same construction as before, BK = HP, being the halves of equals (20, *Cor.*), therefore the triangles CKB and CPH are equal (59, *Cor.* i.) and CK = CP.

Theorem.

63. *The greater of two unequal chords is nearer the centre than the less, and reciprocally.*

1°. Let the chord AB (Fig. 54) be greater than the chord HT, then $CK < CV$. Draw the chord HD = AB, then the arc HD = arc AB > arc HT (23, 50); draw also the perpendicular CP, and CP = CK (62). But $CV > CO$, and $CO > CP$ (48), therefore $CV > CP$, and consequently $CK < CV$.

We have supposed the line CP to lie outside the triangle CVH, because it cannot lie inside (60), nor can it fall on CV, else two perpendiculars can be drawn from the point H to CV.

2°. If $CK < CV$, then $AB > HT$. For, by the first part of the theorem, AB cannot be less than HT, and (62) it

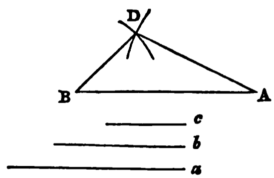
cannot be equal to it, therefore AB must be greater than HT.

PROBLEMS.

64. I. *To construct a triangle having for its sides three given straight lines.*

Draw a line AB = a, one of the given lines (Fig. 55). With A as centre and b, another of the given lines, as radius, describe an arc of a circle, and with B as centre and the remaining line as radius describe another arc of a circle. Join B and A with D, the point of intersection of the two arcs, and BDA is the triangle required.

Fig. 55.



That the construction be *possible*, it is evident that the two arcs must intersect, hence it is necessary that each of the given lines be less than the sum of the other two.

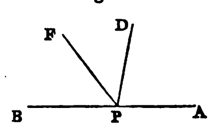
If the two arcs be described with the same length of radius, the triangle will be isosceles; and if the radius be equal to BA, it will be equilateral. Hence, *an equilateral triangle may be constructed on any given line.*

If the three sides of a given triangle be taken as the three given lines, another triangle may be constructed equal to the given triangle (57).

II. *Two anlges of a triangle being given, to find the third.*

The third angle is supplement to the sum of the other two (46, *Cor.* ii.); hence, at any point P in the indefinite line AB (Fig. 56) make an angle APD equal to one of the given angles (43, III.), and at the same point make an angle DPF equal to the other given angle; then, FPB is the required angle;

Fig. 56.



for it is the supplement of FPD + DPA (9), and, therefore, the supplement of the sum of the two given angles.

An angle is said to be *given* when it is represented on a plane surface, such as a board, or a sheet of paper; or when sufficient *data* are given for so representing it. Hence, though the number of degrees in an angle be known, the angle may or may not be *given* in the sense in which the word is here employed. Elementary geometry enables us in many particular cases to represent angles when the number of degrees which each contains is known, as, for instance, an angle of 90° (13, I., 29, I.), of 45° (43), of 60° (52), of 30°, &c., but does not supply a general method applicable to angles of every magnitude.

III. *A side and the two contiguous angles of a triangle being given, to construct the triangle.*

Draw a line equal to the given side. At one extremity make an angle with it equal to one of the given angles; and at the other extremity, and on the same side of the line, make an angle equal to the other given angle. The two sides, one belonging to each of the angles, when produced to meet, will form the required triangle (55).

That the construction be *possible*, it is evident that the sum of the two given angles must be less than 180°.

IV. *To inscribe a circle in a given triangle.*

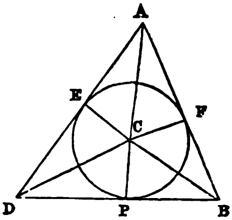
When each of the sides of a rectilinear figure is a tangent to the same circle, the circle is said to be *inscribed* in the figure; and the figure is said to be *circumscribed* about the circle.

When the circumference of the circle passes through the vertex of each of the angles, the figure is said to be *inscribed* in the circle.

Sol. Let ABD (Fig. 57) be the given triangle. Bisect

(43) the angles ABD and BAD, by the straight lines BC and AC. Draw CP perpendicular to BD, CF perpendicular to AB, and CE perpendicular to AD. Then the triangle BCF = BCP. Because CFB = 90° = CPB, CBF = CBP, and BC opposite the right angles, is common to the two triangles. Hence (59, *Cor.* ii.) CF = CP. Also the triangle ACF = ACE, because AFC = AEC, CAF = CAE, and AC is common to the two triangles. Hence (59, *Cor.* ii.) CE = CF = CP. Therefore if a circle be described with C as centre and CP as radius, the sides of the given triangle will be tangents to the circle at P, E, and F (28).

Fig. 57.



If C and D be joined, the triangles DCE and DCP will be equal (59, *Cor.* i.), hence CDE = CDP.

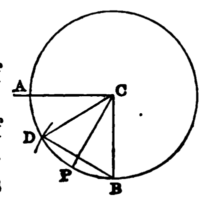
The three lines CA, CB, and CD which bisect the three angles of the triangle are called the *bisectors* of the angles. Hence,

The bisectors of the three angles of a triangle must meet in a point.

V. *To trisect a right angle.*

Let ACB (Fig. 58) be a right angle. With the vertex of the angle as centre and any radius, describe a circle, and with either point of intersection of the circle with one of the sides as centre and the same radius, describe an arc cutting the circumference of the circle in the point D. Join D and B, D and C. Then the arc DB is an arc of 60° (52); consequently the central angle DCB is an angle of 60°, and DCA is an angle of 30°. Hence if a line CP be drawn bisecting the angle DCB (43), the two lines CP and CD will trisect the right angle.

Fig. 58.



Elementary geometry, in which the straight line and circle only are admissible, does not supply any solution of the general problem "*to trisect an angle.*" In the application of algebra to geometry, however, several curves are met with, which by their intersections afford sufficient *data* for the solution.

THE QUADRILATERAL.

65. *Definitions.* A *quadrilateral* is a figure of four sides.

A *parallelogram* is a quadrilateral in which the opposite sides are parallel.

When a parallelogram contains a right angle, it is called a *rectangle;* and a rectangle having two contiguous sides equal is called a *square.* A *trapezium* is a quadrilateral in which only two of the sides are parallel.

A quadrilateral having its four sides equal and containing no right angle is called a *rhombus* or *lozenge.*

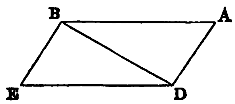
A *diagonal* of a figure is a straight line joining the vertices of any two angles of the figure which are not adjacent, that is, which are not formed at the same side of the figure.

THEOREM.

66. *A diagonal divides a parallelogram into two equal triangles, and the opposite sides and angles of every parallelogram are equal.*

Let ABED (Fig. 59) be a parallelogram in which AB is parallel to ED, and BE parallel to AD. Draw the diagonal BD, then the angle ABD = BDE, and BDA = DBE (34), and the side BD is common to the two triangles ABD and BDE; therefore they are equal (55). The side AD = BE (59, *Cor.* ii.), and AB = DE. Also the angle A = E, and ABD + DBE = ABE = BDE + BDA = EDA.

Fig. 59.



Cor. i. A triangle may always be completed into a parallelogram double of itself by drawing from the vertices of any two of the angles lines parallel to the opposite sides. These lines, when produced to meet, will form a parallelogram of which the third side of the triangle will be a diagonal.

Cor. ii. *Parallel lines between parallels are equal;* for, taking them in pairs, if AD and BE (Fig. 59) be two parallel lines intersecting two others AB and DE, they will form with them a parallelogram (65), the opposite sides of which are necessarily equal.

Cor. iii. *The four sides of a square are equal.*

Theorem.

67. *A quadrilateral in which the opposite sides are equal, two and two, or in which any two opposite sides are equal and parallel, is a parallelogram.*

1°. Let ABED (Fig. 59) be a quadrilateral in which AB = DE, and AD = BE, then ABED is a parallelogram.

Draw the diagonal BD, and the two triangles ABD and BDE are equal (57). Therefore the angle ABD = the angle BDE, and ADB = angle DBE; consequently the sides AB and DE, as also the sides AD and BE are parallel (35), that is, ABED is a parallelogram.

2°. If AB be parallel and equal to DE, then AD will be equal and parallel to BE, and the figure will be a parallelogram. Draw BD, and the triangle ABD is equal to the triangle BDE; for, by supposition, AB = DE, BD is common to the two triangles, and the angle ABD = BDE, since AB and DE are parallel. Therefore, the angle ADB = DBE, and AD is parallel to BE (35). Consequently ABED is a parallelogram.

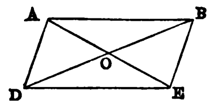
Cor. i. *Hence a rhombus is a parallelogram.*

THEOREM.

68. *The diagonals of a parallelogram bisect each other.*

Let AE and BD (Fig. 60) be the two diagonals of the parallelogram ABED, then AO = EO, and BO = DO. Because the two triangles AOB and DOE are equal (55), having AB = DE (66), the angle ABO = ODE, and OAB = OED (34). Therefore AO = OE (59, *Cor.* ii.), and BO = OD.

Fig. 60.



Cor. If AD = DE, the two triangles DAO and DEO are equal (57), therefore the angle DOA = DOE, that is, DO is perpendicular to AE (13). Hence, in a square, and in a rhombus, the two diagonals bisect each other perpendicularly.

THEOREM.

69. *The sum of the four angles of every quadrilateral is four right angles, or* 360°.

A diagonal divides a quadrilateral into two triangles, and the sum of the four angles of the quadrilateral is the same as the sum of the six angles of the two triangles, viz., twice two right angles, or 360° (46).

Cor. The sum of two adjacent interior angles of a parallelogram is two right angles. Because, since the opposite angles are equal (66), *twice* the sum of any two adjacent interior angles is equal to the sum of the four angles of the parallelogram, or 360°. Therefore the sum of two adjacent interior angles is 180°, or two right angles. Hence

All the angles of a rectangle are right angles.

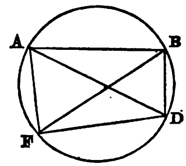
A square has its four sides equal and its four angles right angles (65, 66, *Cor.* iii.).

THEOREM.

70. *The sum of two opposite angles of a quadrilateral inscribed in a circle is two right angles.*

The quadrilateral ABDF (Fig. 61) being inscribed, the angle B is measured by half the arc AFD (40), and the opposite angle F by half the arc ABD (40). The sum of their measures, therefore, is half the circumference, the measure of the sum of two right angles. In like manner A is measured by half the arc FDB, and D by half the arc BAF.

Fig. 61.



This would not be the case if the quadrilateral were not inscribed; for if the angle B, for instance, were inside or outside the circumference of the circle, its measure would be greater or less than half the arc AFD (41, 42).

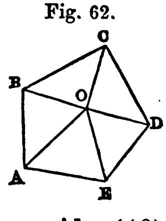
POLYGONS IN GENERAL.

71. Every rectilinear figure is a *polygon*. A polygon is said to be *regular* when it has all its sides equal, and all its angles equal. If either of these conditions is wanting, the polygon is *irregular*.

THEOREM.

72. *The sum of all the interior angles of any polygon is equal to as many times two right angles as the figure has sides, less four right angles.*

Fig. 62.



Straight lines drawn from any point O within the polygon ABCDE (Fig. 62) to the vertices of the angles form as many triangles as the figure has sides. The sum of the interior angles of the polygon is equal to the sum of the angles of all the triangles less the sum of the angles formed at the point O; that is, to as many times two right angles as there are sides (46), less four right angles (9, *Cor.* i.).

Cor. i. *The sides of a polygon being produced in order, the sum of the exterior angles is four right angles, or* 360°, *whatever be the number of sides.* For the sum of every exterior and its adjacent interior angle is two right angles (9). Taking then n for the number of sides, and r for one right angle, the sum of all the exterior and interior angles of any polygon is $2nr$. But the sum of the interior angles is $2nr - 4r$, which, when subtracted from $2nr$, leaves for the sum of the exterior angles $2nr - (2nr - 4r) = 4r$, or 360°.

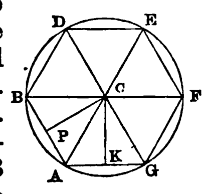
Cor. ii. As the angles of a *regular* polygon are all equal, the magnitude or measure of the common angle a of a regular polygon will be $a = \frac{2nr - 4r}{n} = \frac{(2n - 4)r}{n}$. This formula gives for the common angle of an equilateral triangle $\frac{(2 \times 3 - 4)90^\circ}{3} = 60^\circ$. For the common angle of a pentagon $\frac{(2 \times 5 - 4)90^\circ}{5} = 108^\circ$. For the common angle of a regular hexagon $\frac{(12 - 4)90^\circ}{6} = 120^\circ$; for a regular octagon, 135°, &c.

THEOREM.

73. *A circle can be described about, and inscribed in, any given regular polygon.*

1°. Let ABDEFG (Fig. 63) be a regular polygon. Bisect any two adjacent angles, as ABD and BDE, and the bisectors must meet in some point C (35, *note*), making the triangle CBD isosceles (48), the angles CBD and CDB being the halves of equal angles (71). Join C and the vertices of the other angles. Then the triangle CDB is equal to the triangle CDE (54), for CD is common, DB = DE, and the angle CDB = CDE, since BDE has been bisected. Therefore CE = CB = CD, and as the angle CED = CDE, CE bisects the angle DEF. By a

Fig. 63.



similar demonstration it may be shown that CE = CF = CG, &c. If, therefore, a circle be described with C as centre and CB as radius the circumference will pass through the vertex of each of the angles; that is, the circle will be described about the given polygon.

2°. When a circle is described about a regular polygon the sides of the polygon are equal chords. Therefore the perpendiculars CP, CK, &c., from the centre to these equal chords are equal (62). Hence if with C as centre and CP, one of the perpendiculars, as radius, a circle be drawn, the sides of the polygon will be tangents to it (28), and the circle will be inscribed in the polygon.

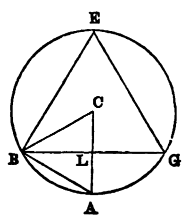
The radii of the inscribed circle are called the *right radii* of the polygon, and the radii of the circumscribed circle are called the *oblique radii* of the polygon.

Cor. From the demonstration it follows that the bisectors of the angles of a regular polygon must all meet in a point.

THEOREM.

74. *The oblique radius of an equilateral triangle is double of the right radius.*

Fig. 64.



A circle being described about the equilateral triangle BEG (Fig. 64), the right radius CL produced to A bisects the chord BG and the arc BAG (20, *Cor.* 24). Therefore since arc BAG is one-third of the entire circumference, the arc BA = 60°, and the chord BA = BC (52); consequently CL = LA (20) $= \frac{CA}{2} = \frac{CB}{2}$.

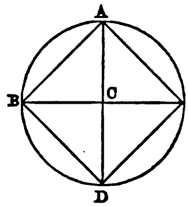
Cor. i. Since the angle BCL is an angle of 60°, and CLB is a right angle, therefore CBL is an angle of 30°. Hence in a right-angled triangle which has one of its acute angles of 60° and the other of 30°, the side opposite the angle of 30° is half the hypothenuse.

Problems.

75. I. *To inscribe a square in a given circle.*

Let ABD (Fig. 66) be the given circle. Draw two diameters AD and BH so as to be perpendicular to each other (29, I.), and join their extremities by straight lines. Then AB = BD = DH = HA (3, ii.). The equilateral figure ABDH, therefore, is a parallelogram (67); and, since the angle BAH is a right angle (40, iv.), consequently it is a square (65).

Fig. 66.



Cor. i. If tangents be drawn to the circle at the points A, B, D, and H, they will form another square which will be circumscribed about the given circle.

Cor. ii. The chord of half the arc BD will be the side of a regular octagon; and, in like manner, a regular polygon of 16, 32, 64, &c., sides may be inscribed in the given circle.

II. *To inscribe a regular hexagon and an equilateral triangle in a given circle.*

Let ABF (Fig. 63) be the given circle. Take any point A in the circumference as centre, and describe a circle with the radius of the given circle as radius. Let B be one of the points of intersection of the two circumferences, and draw AB. Then six chords in succession, each equal to AB, subtend the entire circumference (52), and form an equilateral inscribed hexagon. It is also equiangular; for joining the centre of the circle and the vertices of the angles, the triangles thus formed are all equilateral, and each of the angles ABD, BDE, &c., is equal to the sum of two angles of an equilateral triangle, or 120°.

Straight lines joining the vertices of the angles of the regular hexagon taken alternately, as A, F, and D, will form an inscribed equilateral triangle.

The side of a regular inscribed polygon under six sides is longer than the radius of the circle: it is shorter in a regular polygon above six sides.

As the number of sides in a regular inscribed polygon increases, the right radius approaches towards an equality with the oblique radius; because the sides, becoming shorter chords, recede from the centre. But the right radius cannot reach an equality with the oblique one until the number of sides in the polygon becomes indefinite; in which case, the sides of the polygon become indefinitely small, coinciding sensibly with the arcs they subtend, and the regular polygon becomes sensibly the circle itself.

There are but three regular polygons the angles of which joined together fill exactly a space, viz., six equilateral triangles, four squares, three regular hexagons, the sum of the angles being 360°. But a space may be filled by the angles of some regular polygons of a different number of sides, such as two equilateral triangles and two regular hexagons. A space may also be filled by innumerable irregular ones.

Exercises.

1. Two angles which have their two sides perpendicular, each to each, are equal or supplementary.

2. The extremities of the base of an isosceles triangle are equally distant from the opposite sides; and the lines which bisect the angles at the base and meet the opposite sides are equal.

3. A line drawn from the vertex of any angle of a triangle to the middle point of the opposite side is less than half the sum of the two sides containing the angle.

4. A point equidistant from two straight lines which meet is in the bisector of the angle which these lines contain, and conversely.

5. In a right-angled triangle the hypothenuse is equal to twice the line joining its middle point and the vertex of the right angle.

6. Any angle of a triangle is equal to, less, or greater than a right angle according as the line drawn from the vertex of the angle to the middle point of the opposite side is equal to, greater or less than half the opposite side.

7. The shortest chord that can be drawn through a given point within a circle is a chord at right angles to a line joining the point and the centre of the circle.

8. Two equal chords intersect each other at a given point within a circle, prove that the two segments of the one are respectively equal to the two segments of the other.

9. If two lines be drawn from a point within a circle which is not the centre to the circumference, making unequal angles with a diameter passing through the point, the line which makes the smaller angle with the diameter will be longer than the other. Will the same be true if the point be outside the circle?

10. Two sides and the contained angle of a triangle being given, to construct it.

11. Two sides of a triangle being given, and an angle opposite one of the sides, to construct it

1°. When the given angle is a right angle or an obtuse angle.

2°. When it is acute and opposite the shorter side. When is the solution impossible?

12. To trisect an angle of 45°.

13. If two parallelograms have two adjacent sides respectively equal, but the angles which these sides contain unequal, a diagonal drawn from the vertex of the contained angle is longer in the parallelogram whose sides contain the smaller angle than in the other.

14. When the diagonals of a parallelogram are equal it is a rectangle, when they are equal and perpendicular it is a square, and when they are perpendicular but unequal it is a rhombus.

15. If the opposite angles of a quadrilateral figure are equal it is a parallelogram.

16. A side and the two diagonals of a parallelogram being given, to construct it.

17. To construct a square and a regular hexagon on a given line.

18. The sum of the diagonals of a quadrilateral is less than the sum of four lines drawn to the vertices of the angles from any point within the figure different from the point of intersection of the diagonals.

19. If the exterior angle of a regular polygon is an angle of 30°, how many sides has the figure?

20. If two straight lines bisect each other, lines joining their extremities will form a parallelogram.

21. To place a line of given length between two lines which intersect so as to be parallel to another given line.

22. A circle can be described about any quadrilateral in which the sum of two opposite angles is 180°.

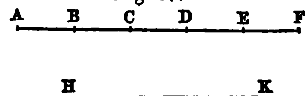
23. When a quadrilateral is described about a circle the sum of one pair of opposite sides is equal to the sum of the other pair.

SECTION III.

PROPORTIONAL LINES.

76. When a line AB (Fig. 67) has been added to itself several times in succession, the line AF obtained may be called the *product* of AB by an abstract number n which expresses how often AB has been employed in the operation. Also any line HK may be *divided* by an abstract number, or by another line AB, the quotient in the former case being a line, in the latter an abstract number expressing how often AB has been *subtracted* from HK so as to leave either no remainder, or a remainder less than AB. As in arithmetic, a line which is contained in another a certain number of times exactly or without remainder, is called a *measure* of the other. When the same line measures each of two others it is called their *common measure*; and if it is the longest line measuring the other two, it is called their *greatest common measure*. Two lines which have a common measure are said to be *commensurable*.

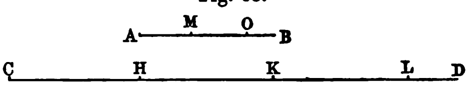
Fig. 67.



I. The geometric method of finding the greatest common measure of two lines is analogous to the arithmetical method of finding the greatest common measure of two numbers.

Thus let AB and CD be two lines, CD being the greater (Fig. 68). With C as centre and AB as radius describe a circle cutting CD in H. With the same radius and H as centre describe another circle cutting CD in K, and so on, until a remainder LD < AB is found. Let the number of equal parts CH, HK, ... be n, then CD = n × AB + LD. Again let AB be divided into m parts, each equal to LD, with a remainder OB; then AB = m × LD + OB. Lastly, let LD contain OB r times without a

Fig. 68.



remainder, then OB is the greatest common measure of CD and AB. For, in the first place, since

$$CD = n \times AB + LD,$$

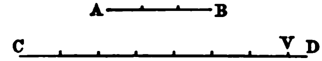
$$AB = m \times LD + OB,$$

$$LD = r \times OB,$$

therefore, OB is contained r times in LD, $r \times m + 1$ times in AB, and $n \times m \times r + n + r$ times in CD; consequently OB is a common measure of AB and CD. Secondly, it is the greatest common measure; for every measure of AB must be a measure of $n \times$ AB or CL; that is, when parts are cut off the line CD, each equal to any common measure of AB and CD, one of the points of division must coincide with L; therefore LD, the remainder of CD, must be a multiple of, or contain a certain number of times exactly, any line which is a measure of AB and CD. For the same reason OB must be a multiple of every common measure of LD and AB, and therefore of every common measure of AB and CD; but OB cannot be a multiple of a line longer than itself; therefore OB is the greatest common measure of AB and CD.

II. In the foregoing we have supposed that a divisor may be found which is contained a certain number of times without remainder in the preceding divisor; and, by continuing the process far enough, this will always be the case when the two lines have any common measure whatever. But *incommensurable* lines, or such as have no common measure, frequently occur, and with these the above process will never produce a remainder equal to zero, no matter how far it may be carried. In such cases, however, it is always easy to find a third line commensurable with one of the two given lines, and differing from the other by a length less than any length which it is possible to name.

Fig. 69.



Thus, for instance, AB and CD (Fig. 69) being two incommensurable lines, let AB be divided into n equal parts; then one of these parts will be contained a certain number of

times exactly in CV, leaving a remainder VD, which is less than the nth part of AB.

Now, since AB is a line of definite length, by greatly increasing n, the length of the nth part of AB is very much diminished, but is always greater than VD. Hence by making the point V approach indefinitely near to D, a line CV is obtained which is commensurable with AB and differs from CD by a length less than any that can be named.

The line CD is called the *limit* of CV, because, by increasing n, CV approaches nearer and nearer to an equality with CD, though the two lines can never become rigorously equal.

The use of this principle will enable us to extend to all kinds of geometrical quantities the algebraic definitions of *ratio* and *proportion* which, in strictness, apply only to such as are commensurable. The subjoined definition of Euclid, though in many respects objectionable, and not so much a definition as a test of proportionals, is yet the only one hitherto devised that applies equally to commensurable and incommensurable quantities; "the first of four magnitudes is said to have the same ratio to the second, that the third has to the fourth, when any equi-multiples whatsoever of the first and third being taken, and any equimultiples whatsoever of the second and fourth; if the multiple of the first be less than that of the second, the multiple of the third is also less than that of the fourth; or, if the multiple of the first be equal to that of the second, the multiple of the third is also equal to that of the fourth; or, if the multiple of the first be greater than that of the second, the multiple of the third is also greater than that of the fourth."

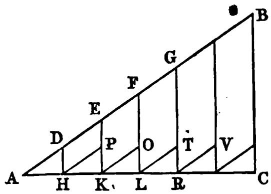
THEOREM.

77. *If lines which intersect two sides of a triangle and are parallel to the third divide one of the sides into equal parts, they will also divide the other side which they intersect into equal parts.*

Let ABC (Fig. 70) be a triangle, take AH = HK = KL .... and draw HD, KE, LF .... parallel to BC, then AD = DE = EF, &c.

Through the points H, K, L, ... draw the lines HP, KO, LT, ... parallel to AB, then the triangles AHD, HKP, KLO . . . . are all equal; for AH = HK = KL . . . . by construction, the angle HAD = KHP = LKO . . . . (34), and the angle AHD = HKP = KLO . . . . (34). Therefore AD = HP = KO = LT, &c. But HP = DE, KO = EF, LT = FG . . . . (66); consequently AD = DE = EF = FG . . . .

Fig. 70.



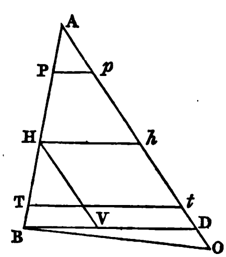
THEOREM.

78. *A line parallel to one of the sides of a triangle divides the other two into proportional parts, and conversely.*

Let Hh (Fig. 71) be parallel to BD, then AH : HB : : Ah : hD.

1°. If AH and HB are commensurable, let AP be equal to their common measure, and mark off parts on AB equal to AP; then one of the points of division must coincide with H. Let the common measure be contained n times in AH, and m times in HB, then AH = $n \times$ AP, HB = $m \times$ AP. If now through the points of division along AB lines be drawn parallel to BD, Ah will contain n parts each equal to Ap, and hD will contain m such parts (77). But

Fig. 71.



$$\frac{n \times \text{AP}}{m \times \text{AP}} = \frac{n \times \text{A}p}{m \times \text{A}p} = \frac{\text{AH}}{\text{HB}} = \frac{\text{A}h}{h\text{D}};$$

that is,

$$\text{AH} : \text{HB} : : \text{A}h : h\text{D}.$$

2°. If AH and HB are incommensurable, divide AH into N equal parts, and along HB mark off lengths equal to one of these parts; also, since no point of division can exactly

coincide with B, let T be the point of division nearest to B, and draw Tt parallel to BD. Then since HT and AH are commensurable $\frac{HT}{AH} = \frac{ht}{Ah}$.

But
$$HT = HB - TB,$$
and,
$$ht = hD - tD,$$
therefore,
$$\frac{HB}{AH} - \frac{TB}{AH} = \frac{hD}{Ah} - \frac{tD}{Ah}.$$

Now TB and tD are less than the Nth parts of AH and Ah respectively; therefore, by increasing N indefinitely the fractions $\frac{TB}{AH}$ and $\frac{tD}{Ah}$ approach indefinitely near to zero; consequently $\frac{HB}{AH} = \frac{hD}{Ah}$, or, taking their reciprocals and writing in the form of a proportion, AH : HB : : Ah : hD.

3°. If AH : HB : : Ah : hD, then Hh is parallel to BD; for, if not, through B a line Bo may be drawn parallel to Hh, and then by the first part of the theorem, AH : HB : : Ah : ho; but, by the hypothesis, AH : HB : : Ah : hD, therefore hD = ho; that is, the line through B parallel to Hh coincides with BD.

Cor. i. Since
$$AH : HB :: Ah : hD,$$
therefore
$$AH + HB : AH :: Ah + hD : Ah;$$
that is,
$$AB : AH :: AD : Ah.$$
Also,
$$AB : AD :: AH : Ah :: HB : hD.$$

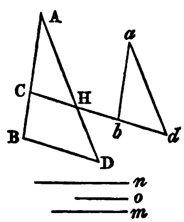
Cor. ii. Drawing HV parallel to AD, Hh = VD (66), and
$$BA : HA :: BD : Hh.$$
Therefore
$$AB : AH :: AD : Ah :: BD : Hh.$$

Problems.

79. I. *To find a fourth proportional to three given lines.*

Let m, n, o, (Fig. 72), be the three given lines. Take any two of them, for instance, n and o, and place them in a straight line AD, so that AH $= n$, HD $= o$. With AD and the third line m make any angle CAD; also let AC $= m$. Draw CH, and through D draw DB parallel to CH, meeting AC produced in B. Then CB is the required line. For (78) AH : HD : : AC : CB; that is, $n : o : : m :$ CB.

Fig. 72.



If $m = o$, then CB is a *third* proportional to the two given lines n and m.

II. *To divide a given line into any given number of equal parts.*

Let AC (Fig. 70) be the given line which it is required to divide into n equal parts. Take a line AB of indefinite length, making any angle BAC with the given line, and mark off n equal parts on AB, using any length of radius. Join the last point of division and the other extremity C of the given line, and through the other points of division along AB draw the lines DH, EK, FL . . . . parallel to BC, these lines will divide the given line into n equal parts (77).

SIMILAR TRIANGLES.

80. *Def.* Two triangles are *similar* which have their three angles equal, each to each.

Equal triangles are similar; but triangles may be similar without being equal (59, *Cor.* iii.).

The sides opposite to equal angles in different triangles are called *homologous* sides.

THEOREM.

81. *Two triangles are similar which have two angles equal, each to each.*

Any angle of a triangle is supplement to the sum of the other two, because the sum of the three is 180° (46). Consequently two triangles, which have two angles equal each to each, have their three angles respectively equal, and are, therefore, similar.

Cor. Two triangles are similar when they have an equal angle each besides a common angle. Also, when they have a common angle, or vertically opposite angles, at the vertex and parallel bases (34).

THEOREM.

82. *Two triangles are similar which have their three sides respectively parallel.*

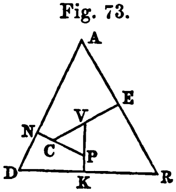
Suppose AB parallel to *ab* (Fig. 72), AD to *ad*, and BD to *bd*. Produce one of the sides, as *db*, to meet AD and AB, or AD and AB when sufficiently produced (30, II.). The angle d = CHA = D, and *abd* = ACH = B (34). Then also A = a.

The parallel sides are the homologous sides.

THEOREM.

83. *Two triangles are similar which have their three sides respectively perpendicular.*

Fig. 73.



Let ADR (Fig. 73) and PCV be two triangles having KV perpendicular to DR, CE perpendicular to AR, and PN perpendicular to AD. Then the triangle ADR is similar to PCV.

The four angles of the quadrilateral DNPK are together equal to 360° (69). By supposition DNP = 90°, and DKP = 90°. Therefore NDK

+ NPK = 180°. But VPC + NPK = 180° (9). Therefore VPC = NDK. For the same reason DAR = PCV, and ARD = PVC; consequently the two triangles are similar.

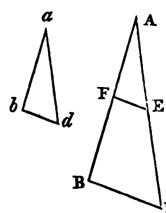
The sides perpendicular to each other are the homologous sides.

THEOREM.

84. *Two triangles are similar which have an equal angle each contained by proportional sides.*

Suppose the angle A = *a* (Fig. 74) and AB : *ab* : : AD : *ad*. Take AF = *ab*, and draw FE parallel to BD. Then AB : AF = *ab* : : AD : AE (78, *Cor.* i.). The three first terms of the two proportions are equal, therefore *ad* = AE ; and the two triangles AFE and *abd* are equal (54). But the triangle ABD is similar to AFE (81, *Cor.*), therefore it is similar to *abd*.

Fig. 74.



THEOREM.

85. *Two triangles are similar which have their three sides respectively proportional.*

Suppose AB : *ab* : : AD : *ad* : : BD : *bd* (Fig. 74). Take on AB a part AF = *ab*, and draw FE parallel to BD. Then AB : AF = *ab* : : AD : AE : : BD : FE (78, *Cor.* i., ii.). In these two proportions the antecedents of the corresponding ratios are equal, and also the consequents of the first ratios; therefore AE = *ad*, and FE = *bd*. Hence the triangle AFE is equal to the triangle *abd* (57); but the triangle AFE is similar to ABD (81, *Cor.*), so also, therefore, is the triangle *abd*.

THEOREM.

86. *The homologous sides of two similar triangles are proportional.*

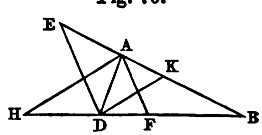
Let ABD and *abd* (Fig. 74) be two similar triangles in which A = *a*, B = *b*, D = *d*. Take AF = *ab*, and draw FE parallel to BD. Then the triangle AFE is equal to the triangle *abd*, because A = *a*, B = *b* = AFE, and AF = *ab*, by construction. Therefore (59, *Cor.* ii.) AE = *ad*, and FE = *bd*. But since FE is parallel to BD, we have AF = *ab* : AB : : AE = *ad* : AD : : FE = *bd* : BD (78, *Cor.* i., ii.); or *ab* : AB : : *ad* : AD : : *bd* : BD, the proportion of the homologous sides.

PROPERTIES OF TRIANGLES FROM PROPORTIONAL LINES.

87. *A line bisecting any angle of a triangle divides the opposite side into segments which are related to each other as the contiguous sides.*

Let AF (Fig. 75) bisect the angle A in the triangle BAD. Draw from either of the other two angles, D for instance, DE parallel to AF, and produce BA to meet that parallel at E. Then BAF and BED are similar triangles (81, *Cor.*); and the triangle DAE is isosceles, because the angle E = BAF = FAD = ADE (34). Therefore AD = AE (48), and BF : FD : : BA : AE = AD (78).

Fig. 75.



Cor. Let AH bisect the exterior angle DAE, and draw DK parallel to AH. Then, as before, KAD is an isosceles triangle, and BH : HD : : BA : AK = AD (78, *Cor.* i.).

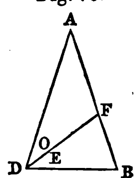
BD is said to be cut *internally* in F, and *externally* in H.

THEOREM.

88. *In an isosceles triangle which has each of the angles at the base double of the angle at the vertex, a line bisecting either of the angles at the base divides the opposite side into mean and extreme ratio; that is, the greater segment is a mean proportional between the entire side and the smaller segment.*

Let the isosceles triangle BAD (Fig. 76) have the angle B = D = 2A; and let DF bisect the angle D, so that E = O = A. Then the angle BFD = A + O = B (47); and BD = DF = AF (48). Therefore the two triangles BAD and BDF are similar (81), and BF : BD : : BD : AB (86); or BF : AF : : AF : AB.

Fig. 76.

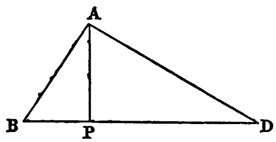


THEOREM.

89. *A line from the vertex of a right-angled triangle perpendicular to the hypothenuse divides the right-angled triangle into two small ones similar to the great one and to each other.*

The line AP (Fig. 77), from the right angle A of the triangle BAD, perpendicular on the hypothenuse BD, divides the triangle into two right-angled triangles BPA, DPA, each of which has a right angle at P, and an angle common with the great triangle, one at B, the other at D. Therefore the three triangles are similar (81, *Cor.*)

Fig. 77.



Cor. The perpendicular from the right angle is mean proportional between the two segments of the hypothenuse; and each of the small sides is mean proportional between the hypothenuse and the contiguous segment.

The two similar triangles DPA, BPA (Fig. 77) give

$\div$ DP : AP : BP; the two DPA, DAB give $\div$ DP : DA : DB; and the two BPA, BAD give $\div$ BP : AB : BD.

THEOREM.

90. *In a right-angled triangle the square of the hypothenuse is equal to the sum of the squares of the two small sides.*

Since, by the last *Cor.*, AD and AB (Fig. 77) are mean proportionals between the hypothenuse and the contiguous segments, therefore $AD^2 = DP \times BD$, and $AB^2 = BP \times BD$. Hence $\overline{DP + BP} \times BD = BD \times BD = BD^2 = AD^2 + AB^2$.

By the *square* of a line, it must be remembered, is here meant nothing more than the second power of the number which expresses how often the line contains the *unit of length.* Thus (Fig. 77) if $AB = 3$ inches, and $AD = 4$ inches, then $BD^2 = 3^2 + 4^2 = 25$; and therefore $BD = 5$ inches.

The *product* of a line by a line is to be understood in a similar sense.

A more extended meaning will be given to these expressions hereafter.

Cor. i. *The square of each small side of a right-angled triangle is equal to the difference of the squares of the hypothenuse and of the other small side.*

Cor. ii. *The difference of the squares of the two small sides is equal to the difference of the squares of the two segments of the hypothenuse.*

The two small sides AD and AB (Fig. 77) being the hypothenuses of the two right-angled triangles APD, APB, therefore $AD^2 = DP^2 + AP^2$, and $AB^2 = BP^2 + AP^2$. Consequently $AD^2 - AB^2 = DP^2 - BP^2$.

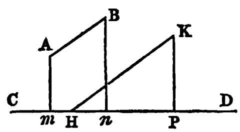
Cor. iii. *In an isosceles right-angled triangle the square of the hypothenuse is double of the square of either of the two small sides.*

THEOREM.

91. *In any triangle the square of a side opposite an acute angle is equal to the sum of the squares of the other two sides, minus twice the product of one of these sides by the projection on it of the other.*

Def. The *projection* of one line on another is the part of the latter intercepted between two perpendiculars drawn to it from the extremities of the other line. Thus *mn* (Fig. 78) is the projection of AB on CD, if A*m* and B*n* are perpendicular to CD; and, in like manner, if KP is perpendicular to CD, HP is the projection of HK.

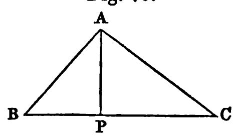
Fig. 78.



A line on which another is projected is supposed to be indefinitely produced.

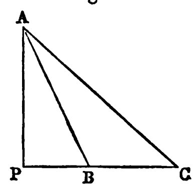
Let BCA (Fig. 79) be an acute angle, and AP perpendicular to BC, or BC produced (Fig. 80). Then $AB^2 = AP^2 + BP^2$ (90). But, if the perpendicular falls inside, $BP^2 = \overline{BC - PC}^2 = BC^2 + PC^2 - 2BC \times PC$. Therefore $AB^2 = AP^2 + BC^2 + PC^2 - 2BC \times PC = AC^2 + BC^2 - 2BC \times PC$, since $AP^2 + PC^2 = AC^2$.

Fig. 79.



If AP falls outside the triangle, then (Fig. 80) $BP^2 = \overline{PC - BC}^2 = PC^2 + BC^2 - 2BC \times PC$. Consequently $AB^2 = AP^2 + BP^2 = AP^2 + PC^2 + BC^2 - 2BC \times PC = AC^2 + BC^2 - 2BC \times PC$.

Fig. 80.



THEOREM.

92. *In any triangle the square of a side opposite an obtuse angle is equal to the sum of the squares of the other two sides, plus twice the product of one of these sides by the projection on it of the other.*

Let ABC (Fig. 80) be an obtuse angle, and a perpendicular from A to the opposite side must fall outside the triangle

(46). Then $AC^2 = AP^2 + PC^2$. But $PC^2 = \overline{PB + BC}^2 = PB^2 + BC^2 + 2BC \times PB$. Therefore $AC^2 = AP^2 + PB^2 + BC^2 + 2BC \times PB = AB^2 + BC^2 + 2BC \times PB$.

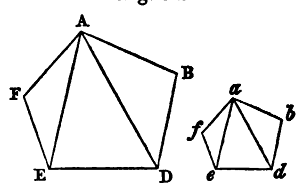
93. Comparing the three last theorems, it is easy to see that according as the square of any side of a triangle is equal to, less, or greater than the sum of the squares of the other two sides, the opposite angle is equal to, less, or greater than a right angle.

PROPERTIES OF POLYGONS FROM PROPORTIONAL LINES.

94. A polygon being given, it is always *possible* to construct another of the same number of sides, having its angles respectively equal to the angles of the given polygon, and the sides containing any angle proportional to the sides containing the equal angle in the given polygon.

Thus, let ABDEF (Fig. 81) be a polygon, draw the diagonals AD, AE, and at the extremities of a line *ab* of any length make the angles *abd*, *bad*, equal respectively to ABD, BAD (43, III.); then the two triangles ABD and *abd* are similar (81).

Fig. 81.



In like manner a triangle *ade*, similar to ADE, may be constructed on the line *ad*, and a triangle *aef*, similar to AEF, may be constructed on the line *ae*. Hence, from the pairs of similar triangles, the angle *b* = B, *bde* = *adb* + *ade* = ADB + ADE = BDE, and so on with the other angles of the polygon.

Also (86) *ab* : AB : : *bd* : BD : : *ad* : AD : : *de* : DE. . . . . Consequently, the sides containing any angle in the polygon *abdef* are proportional to the sides containing the equal angle in the polygon ABDEF.

Def. Two polygons which have their angles respectively equal, and the sides containing equal angles proportional, are said to be *similar*.

In the case of triangles, one of these conditions necessarily involves the other (85, 86).

The equal angles in two similar polygons are called *homologous* angles, and the sides containing equal angles are called *homologous* sides.

Diagonals are homologous which join the vertices of homologous angles; and, in general, two lines are homologous which join homologous points, that is, points taken on homologous sides at distances from homologous angles proportional to the homologous sides.

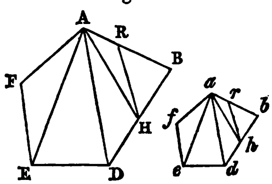
The ratio of any homologous sides in two similar polygons is called the *ratio of similitude* of the polygons.

THEOREM.

95. *Similar polygons have their homologous diagonals and homologous lines proportional to the homologous sides.*

Suppose the two polygons (Fig. 82) to be similar. Draw the homologous diagonals AD and *ad*, AE and *ae*, also the homologous lines AH and *ah*, HR and *hr*. The two triangles ABD and *abd* are similar (84), having, by supposition, the angles $B = b$, contained by proportional sides. Therefore, AB : *ab* : : AD : *ad*, &c. Likewise, the two triangles ADH, *adh*, are similar, having two equal angles at D and *d*, contained by proportional sides; hence AD : *ad* : : AH : *ah* : : DH : *dh* : : BD : *bd*. Again, the two triangles AHR and *ahr* are similar, and AH : *ah* : : HR : *hr* : : BR : *br* : : AB : *ab*, &c.

Fig. 82.



Cor. Since AB : *ab* : : BD : *bd* : : DE : *de* : : EF : *ef* : : FA : *fa* (94, *Def.*), therefore AB + BD + DE + EF + FA : *ab* + *bd* + *de* + *ef* + *fa* : : AB : *ab* : : AD : *ad* : : RH : *rh*; that is, the *perimeters* of similar polygons are proportional to their homologous sides, to their homologous diagonals, to any of their homologous lines.

Any two *regular* polygons of the same number of sides are similar; because, each of the polygons being equilateral, their sides are proportional; and their angles are re-

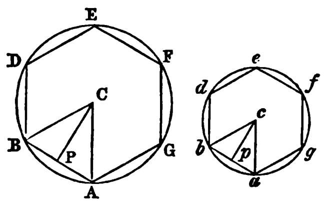
spectively equal, since any angle of a regular polygon is expressed by the formula $a = \frac{(2n-4)r}{n}$, n being the number of sides, and r one right angle (72 *Cor.* ii.).

THEOREM.

96. *The oblique and right radii of similar regular polygons are proportional to the sides.*

Let ABDEFG and *abdefg* (Fig. 83) be similar regular polygons. The oblique radii CB, CA, *cb*, *ca* bisect the equal angles ABD, BAG, *abd*, *bag* (73); therefore, the triangles BCA and *bca* are similar, as are also the triangles BPC, *bpc* formed by drawing the right radii CP and *cp*. Consequently AB : *ab* : : BC : *bc* : : CP : *cp*.

Fig. 83.



Cor. i. The perimeters of similar regular polygons are proportional to the sides, to the right and oblique radii, to any of their homologous lines (95 *Cor.*).

Cor. ii. If a circle be described about each of the polygons in Fig. 83, two other similar regular polygons of double the number of sides will be formed by joining the middle points of the arcs which the sides subtend with the vertices of the contiguous angles; and by conceiving this process as repeated indefinitely, the perimeters of the polygons will approach indefinitely near to an equality with the circumferences of the circumscribed circles, and the right radii to an equality with the radii of the circles. Hence,

The circumferences of unequal circles are proportional to their homologous sides; that is, to their radii, to their diameters, to their homologous chords, to their homologous arcs.

Two arcs are homologous which contain the same number of degrees, and chords of such arcs are homologous chords.

Def. Since by increasing the number of sides in an inscribed polygon, it may be made to differ from the circle by

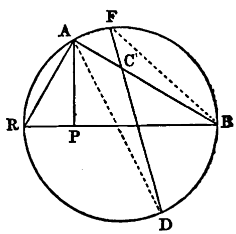
a quantity less than any that can be named, though, for want of a number strictly infinite, the two can never become rigorously equal, the circle is called the *limit* of the polygon; the circumference is the *limit* of the perimeter, and the radius of the circle is the *limit* of the right radius of the polygon.

The same may be applied to the circumscribed polygon.

THEOREM.

97. *The segments of two chords intersecting each other in a circle are reciprocally proportional.*

Fig. 84.



Suppose the two chords AB and FD (Fig. 84), intersecting each other at C. Join their extremities by the lines FB and AD. The two triangles FCB and ACD are similar, on account of the two equal vertical angles at C, and the equal inscribed angles B = D and F = A. Therefore, FC : AC : : BC : DC (86).

THEOREM.

98. *A line from any point of the circumference, perpendicular to a diameter, is mean proportional between the segments of the diameter.*

Let AP (Fig. 84) be perpendicular on the diameter BR. Draw from A two chords AB and AR to the extremities of the diameter. Then BAR is a right-angled triangle, BR its hypothenuse (40, iv.), and AP a perpendicular from the right angle. Hence $\div$ RP : AP : BP (89, *Cor.*).

Also $AP^2 = RP \times BP$.

Cor. The squares of two chords from the same or opposite extremities of a diameter are proportional to the corresponding segments of that diameter; for since

$AR^2 = RP \times RB$, and $AB^2 = BP \times RP$ (89, *Cor.*),

$AR^2 : AB^2 :: RP \times RB : BP \times RB :: RP : BP$;

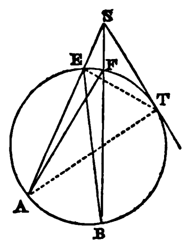
and in like manner it may be proved when the chords are drawn from the same extremity of the diameter.

THEOREM.

99. *Two secants from the same point outside a circle are reciprocally proportional to their exterior parts.*

From the extremities A and B (Fig. 85) of two secants AS and BS meeting at S, draw the two chords AF and BE to their intersections with the circumference, and the two triangles AFS, BES are similar, on account of the common angle S, and the two equal inscribed angles at A and B; therefore AS : BS :: FS : ES (86).

Fig. 85.



Cor. A tangent which meets a secant is mean proportional between the entire secant and its exterior part.

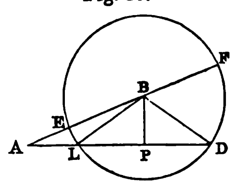
Let the secant BS be conceived to revolve round the fixed point S, making a constantly increasing angle with AS, then B and F will approach each other, and finally coincide at T. In this position BS becomes a tangent (26), AF coincides with AT, EB with ET, and BS = FS = TS; the proportion above therefore becomes AS : BS = TS :: FS = TS : ES.

THEOREM.

100. *A perpendicular from any angle in a triangle to the opposite side divides that side into such segments that the entire side is to the sum of the other two sides, as the difference of the same two sides is to the difference, or to the sum of the two segments, according as the perpendicular falls inside or outside the triangle.*

1°. Let BP be a perpendicular from B to the side AD of the triangle ABD, and let BD be less than BA. Describe a circle with B as centre and BD as radius, and produce AB to meet the circumference in F. Then (99) AD : AF = AB + BD :: AE = AB − BD : AL = AP − PD, since LP = PD (20, *Cor.*).

Fig. 86.



2°. If ABL be the triangle, describe

a circle, with B as centre, and BL as radius, and draw BP perpendicular to AL produced to D. Then (99) AL : AF = AB + BD : : AE = AB − BD : AD = AP + PL.

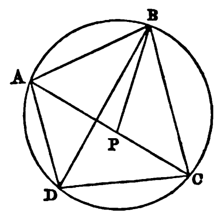
If BD = AB, the third term in the first proportion becomes equal to zero, therefore AP = PD, that is, the perpendicular bisects the side AD; the same result as that already obtained (51).

Theorem.

101. *In every inscribed quadrilateral, the product of the diagonals is equal to the sum of the products of the opposite sides.*

Let ABCD (Fig. 86*a*) be an inscribed quadrilateral, and make the angle CBP equal to the angle ABD. The two triangles ADB and CBP are similar, for the angles ABD and CBP are equal by construction, and the inscribed angle ADB is equal to the inscribed angle ACB (40).

Fig. 86*a*.



Therefore AD : DB : : CP : CB,
and AD × CB = BD × CP.

Also the triangles ABP and DBC are similar, for the angle BAP = BDC (40), and the angle ABP = ABD + DBP = CBP + DBP.

Therefore AB : AP : : DB : DC,
and AB × DC = BD × AP.

Consequently AD × CB + AB × DC = BD × CP + BD × AP = BD × $\overline{\text{CP} + \text{AP}}$ = BD × AC.

Problems.

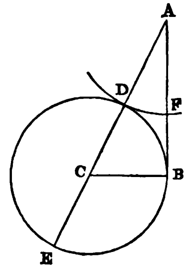
102. I. *To find a line mean proportional between two given lines.*

On a line of indefinite length take RP (Fig. 84) equal to one of the given lines, and PB equal to the other, and with the middle point of RB as centre, and either half of the line as radius, describe a circle. At P erect a perpendicular PA on RB, and AP is the mean proportional required (98).

II. *To divide a given line into mean and extreme ratio, that is, into two such segments that one shall be a mean proportional between the entire line and the other segment.*

Let AB (Fig. 87) be the given line. At one extremity B erect a perpendicular $BC = \frac{AB}{2}$, and with C as centre, and CB as radius, describe a circle. From the other extremity A draw the secant AE through the centre C; and with A as centre, and AD, the exterior part of the secant, as radius, describe an arc cutting AB in F. Then AB is divided into mean and extreme ratio in F.

Fig. 87.



For, since AB is a tangent (28)

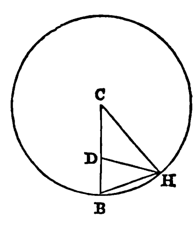
$AB : AD = AF :: AE : AB$ (99, *Cor.*);
therefore $AB - AF = BF : AF :: AE - AB = AF : AB$.

III. *To inscribe in a given circle a regular decagon and a regular pentagon.*

Divide the radius of the given circle into mean and extreme ratio, and a chord equal to its greater segment will be the side of a regular inscribed decagon.

Thus let CB (Fig. 88) be divided into mean and extreme ratio at D. On the smaller segment DB construct an isosceles triangle DHB, so that $BH = DH = CD$ (64, I.). Draw CH, and the two triangles BCH, BHD are isosceles and similar (84), having a common angle at B contained by proportional sides; for $BD : BH = DC :: BH : BC$. Therefore $CH = CB$, and H is a point in the circumference of the circle. Now the angle C is equal to one-half of the angle CHB, or to one-half of CBH, since $C = BHD = CHD$, the triangle CDH being

Fig. 88.



isosceles. Therefore the angle C is one-fifth of the sum of the three angles of the triangle CBH; that is, $C = \frac{180^\circ}{5} = \frac{360^\circ}{10}$. The arc BH, therefore, is one-tenth of the entire circumference, and the chord BH is the side of a regular inscribed decagon.

The chord of an arc double of the arc BH will be the side of a regular inscribed pentagon.

If, from the same point in the circumference, the sides of a regular inscribed decagon and a regular inscribed hexagon be drawn, the arc intercepted between the other extremities of these chords will be an arc of 24°, and its chord will be the side of a regular quindecagon, a polygon of fifteen sides.

Comparing these results with those already obtained (75), we see that a regular polygon may be inscribed in a given circle, when the number of its sides is expressed by any of the terms in the following series:—

$$\begin{array}{rrrrrl} 3, & 6, & 12, & 24, & 48, & \ldots\ldots \\ 4, & 8, & 16, & 32, & 64, & \ldots\ldots \\ 5, & 10, & 20, & 40, & 80, & \ldots\ldots \\ 15, & 30, & 60, & 120, & 240. & \ldots\ldots \end{array}$$

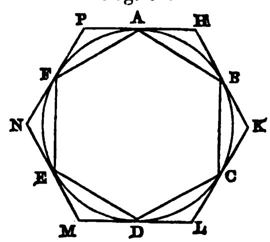
It may also be shown that a regular polygon of $2^n + 1$ sides may be inscribed, when $2^n + 1$ is a prime number; but to inscribe other regular polygons, a method of approximation must be adopted, the principles of elementary geometry being insufficient for the purpose.

When a regular polygon is inscribed in a circle, the vertices of the angles divide the circumference into as many equal parts as there are sides in the polygon. A regular polygon of 360 sides would, therefore, divide the circumference into arcs of 1°; but 360 does not occur in any of the above series, nor is there any elementary method of trisecting the arc of 3° subtended by the side of a regular polygon of 120 sides. Hence, though many multiples of the unit arc are easily determined, the unit arc itself, so important in the construction of mathematical and astronomical instruments, must be determined by approximation.

IV. *A regular polygon being inscribed in a given circle, to circumscribe a similar regular polygon about the same circle.*

At the vertices of the angles of the regular inscribed polygon ABCDEF (Fig. 89), draw the tangents PH, HK, KL, &c., and the circumscribed polygon which they form is similar to the inscribed polygon.

Fig. 89.



For since AF = AB = BC . . . and the angle AFP = FAP = BAH = ABH . . . , all being angles which are measured by the halves of equal arcs (39), therefore the triangles FPA, AHB, BKC . . . . are isosceles and equal. Hence FP = AP = AH = HB . . . , and the angle P = H = K . . . Consequently the circumscribed polygon is regular, and, having the same number of sides, it is similar to the inscribed polygon.

Tangents drawn at the middle points of the arcs AF, AB, &c., will be parallel to the chords, and will also form a regular circumscribed polygon similar to the inscribed one.

The converse problem of inscribing a regular polygon similar to a given circumscribed polygon, can likewise, as is obvious, be solved, either by joining the points of contact, or by drawing chords parallel to the sides of the circumscribed polygon.

V. *To find the ratio of the circumference of a circle to its diameter.*

Of the numerous solutions given of this celebrated problem, one of the simplest depends on the following

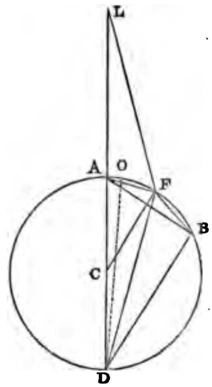
THEOREM.

The radius of a circle being expressed by 1, *if* 2 *be added to the supplemental chord of an arc, the square root of the sum will be equal to the supplemental chord of half the same arc.*

The *supplemental chord* of an arc is the chord of the supplement of the arc.

Take the arc AB (Fig. 90) and bisect it at F, draw the diameter AD, the radius CF, and the chords AF, FB, AB, FD, BD. The chord BD is the supplemental chord of the arc AB; and FD the supplemental chord of the arc AF = $\frac{1}{2}$AB. From F to the prolongation of the diameter AD, draw FL = FD. Then, the isosceles triangles DFL, DCF, are similar, on account of their common angle FDL = DLF = DFC; hence ∺ DC : DF : DL. But the triangles LAF = DBF; because the angles BDF = ADF = ALF, the sides AF = BF, and the angles DBF = LAF, each being supplement to the same angle DAF, as AFBD is an inscribed quadrilateral (70). Therefore, their homologous sides BD = AL. Now, as DL = DA + AL, and DA = 2DC, the above proportion ∺ DC : DF : DL becomes DC : DF :: DF : 2DC + DB. Hence $DF^2 = 2 \times DC^2 + DC \times DB$: and as the radius DC = 1, $DF^2 = 2 \times 1^2 + 1 \times DB = 2 + DB$; and $DF = \sqrt{2 + DB}$; that is, the supplemental chord DF of half the arc AB is equal to the square root of the supplemental chord DB of the entire arc AB increased by 2.

Fig. 90.



If, now, we suppose the arc AB to be an arc of 60°, then the chord AB is equal to the radius (52); and, since ABD is a right-angled triangle (40, iv.), $DB^2 = AD^2 - AB^2 = 2^2 - 1 = 3$; and $BD = \sqrt{3} = 1{\cdot}7320508076$. Adding 2 to this number, and extracting the square root of the sum we get, by the theorem, $DF = \sqrt{3{\cdot}7320508076} = 1{\cdot}9318516525$, where DF is the supplemental chord of an arc of 30° subtended by the side of a regular polygon of 12 sides. When this last number is increased by 2 and the square root extracted, an equation is obtained for the supplemental chord of an arc subtended by the side of a regular polygon of 24 sides. Continuing this process, after the eighth similar extraction of the square root, we obtain $\sqrt{3{\cdot}9999832669}$ as the expression for the supplemental chord of the arc subtended

by the side of a regular polygon of 1536 sides. Such an arc, being less than one-fourth of a degree, may be regarded as sensibly coinciding with its chord, and the ratio of the perimeter of the polygon to its right radius may be taken as the ratio of the circumference of the circumscribed circle to its radius. Let, then, AO be a side of the polygon, and draw its supplemental chord DO. Since AOD is a right angle (40, iv.) $AO^2 = AD^2 - DO^2 = 4 - 3{\cdot}9999832669 = 0{\cdot}0000167331$; and $AO = \sqrt{0{\cdot}0000167331} = 0{\cdot}0040906112$. Therefore the perimeter of the polygon, and, consequently, the circumference of the circumscribed circle, is $0{\cdot}0040906112 \times 1536 = 6{\cdot}2831788$, the radius of the circle being 1. Hence the approximate ratio of the circumference to the diameter is $\dfrac{6.2831788}{2} = 3{\cdot}1415894$.

The circumferences of unequal circles being related to each other as their diameters (96, *Cor.* ii.), the circumference divided by the diameter is the same for every circle, and is represented conventionally by the Greek π. Hence $\pi = 3{\cdot}1415894 = \dfrac{c}{2r}$, or $c = 2\pi r$; from which equation, when the length of the radius is known, the length of the circumference may at once be determined, and conversely.

So also, the radius being 1, the length of an arc of 1° may be found by the proportion $360° : 1° :: 2\pi = 6{\cdot}2831788 : x = 0{\cdot}01745$; and the length of an arc of any number of degrees may be found by multiplying the given number of degrees by $0{\cdot}01745$. Thus the length of an arc of 60° is $0{\cdot}01745 \times 60 = 1{\cdot}047 >$ the radius, which is equal to its chord, by $0{\cdot}047$. The length of an arc of $1'$ is $0{\cdot}01745 \times \dfrac{1}{60} = 0{\cdot}000290$; and the length of an arc of $1''$ is $0{\cdot}01745 \times \dfrac{1}{3600} = 0{\cdot}000005$, or $\dfrac{1}{200000}$ nearly.

The value of π given above must obviously be somewhat too small, as the arc is greater than the side of the polygon which subtends it, whatever be the number of the sides. The ablest geometers tried various mathematical methods to find

the *exact* value of π; but it was proved, more than a hundred years ago, that no such exact value exists, because the circumference and diameter are incommensurable, and, therefore, cannot be exactly expressed in terms of the same unit.

By comparing the perimeters of regular inscribed and circumscribed polygons of 96 sides, Archimedes proved that π is less than $3\frac{10}{70}$, but greater than $3\frac{10}{71}$. He assumed the former, or its equal $\frac{22}{7}$, as sufficiently accurate. Adrian Metius made $\pi = \frac{355}{113}$. Ludolph Van Ceulen determined the value of π to the thirty-second place of decimals. Lagny obtained an expression which is accurate as far as the one hundred and twenty eighth place of decimals; and in recent times the value of π has been accurately expressed as far as the one hundred and fifty-fifth place of decimals.

We shall assume $\pi = 3{\cdot}14159$ as sufficiently accurate for our purposes.

Exercises.

1. To divide a given line similarly to a given divided line, that is, into parts proportional to those of another given line.

2. To cut off the nth part of a given line, n being a given number.

3. Any number of lines which meet in a point are cut proportionally by two parallel lines, and conversely.

4. When the points of bisection of the three sides of a triangle are joined, four triangles are formed which are all equal.

5. In any triangle, lines drawn from the vertices of the angles to bisect the opposite sides must meet in a point.

6. The perpendiculars drawn from the vertices of the angles in any triangle to the opposite sides must all meet in a point.

7. Are two triangles which have two proportional sides and an equal angle each necessarily similar?

8. The bisector of the external vertical angle of a triangle is parallel to the base.

9. If a line drawn from any angle of a triangle divides the opposite side into segments related to each other as the contiguous sides, it is the bisector of the angle.

10. When similar parallelograms have a common angle, the diagonals from the vertex of the common angle must all lie in the same straight line.

11. Perpendiculars drawn from the angles of any triangle to the opposite sides are related to each other inversely as the opposite sides.

12. In a given circle to inscribe a triangle similar to a given triangle.

13. About a given circle to circumscribe a triangle similar to a given triangle.

14. The ratio of two sides of a triangle and the angle which they contain being given, to construct the triangle when the third datum is, 1° the base; 2° the sum or difference of the two sides; 3° the perpendicular from the vertex of the given angle to the base.

15. A perpendicular being drawn from any angle of a triangle to the opposite side, the product of the two sides containing the angle is equal to the product of the perpendicular and the diameter of a circle described about the triangle.

16. If each of the sides of a quadrilateral be bisected, lines joining the points of bisection taken in order will form a parallelogram, having its sides parallel to the diagonals of the quadrilateral.

17. An equilateral triangle being inscribed in a circle, if lines be drawn through the points of bisection of the arcs subtended by the sides, these lines will bisect the sides of the triangle.

18. In a right-angled triangle a perpendicular being drawn from the vertex of the right angle to the hypothenuse, the segments of the hypothenuse are related to each other as the squares of the contiguous sides.

19. The sum of the squares of any two sides of a triangle is equal to twice the square of the line drawn from the vertex of the angle which the sides contain to the middle point of the opposite side, plus twice the square of half the opposite side.

20. The sum of the squares of the four sides of a parallelogram is equal to the sum of the squares of the diagonals.

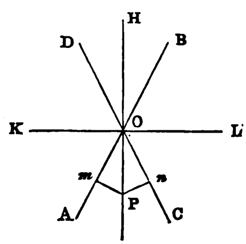
On Loci.

The solution of geometrical problems is often very much facilitated by employing the method of loci, which may be explained briefly as follows:—

1. When a point so moves in a plane as to fulfil, in every position which it assumes, one or more specified conditions, the line which it traces out is called the *locus* of the point. Thus the locus of a point at a given distance from another point which is fixed, is the circumference of a circle of which the fixed point is the centre and the given distance the radius. The locus of a point equidistant from two given points is the perpendicular which bisects the line joining the two given points (15, 16). In the first case, the condition to be fulfilled is, that the moving point be always at a fixed distance from the given point; in the second case the condition is, that its distance from one of the given points be always equal to its distance from the other.

It frequently happens that the moving point can trace out two or more lines, satisfying at the same time the prescribed conditions. In this case, the two or more lines taken together are called the locus. Thus the locus of a point equidistant from two given lines which intersect is the two bisectors of the adjacent angles which the given lines, when produced, form at their point of meeting; for AB and CD (Fig. 91) being two given lines which intersect in O, if HP and KL are drawn to bisect the angles AOC and AOD, and from any point P, in either of the bisectors, perpendiculars Pm, Pn, are drawn to the two given lines, then Pm = Pn, since the triangles POm and POn are equal (59, *Cor.* i.). Therefore a point moving along the lines HP and KL is always equidistant from AB and CD; that is, HP and KL are the locus of a point restricted by the conditions named.

Fig. 91.



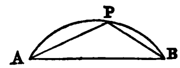
It is also easy to see that no point outside HP and KL can be equidistant from AB and CD. Hence that a line or lines be the locus of a point, it is necessary that the conditions of the point be satisfied by every point in the line or lines, and by no other point.

2. *To find the locus of a point at which a given line subtends a given angle.*

Let AB (Fig. 92) be the given line, and let the lines AP and BP make an angle APB equal to the given angle. Draw through the points A, B, and P,

the circumference of a circle, and lines passing through A and B making an angle equal to APB must meet in the arc APB, or in an equal arc on the other side of AB (40, 41, 42). Hence the complete locus is two equal arcs having AB for a common chord.

Fig. 92.

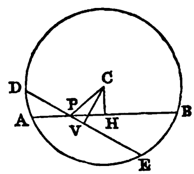


If the given angle be a right angle, the locus will be the circumference of a circle of which AB will be the diameter.

3. *To find the locus of a point which bisects all chords passing through a given point in a given circle.*

Let DE and AB (Fig. 93) be any two chords passing through the given point P in the given circle ADB. Join the centre C and P, and draw the lines CH, CV, bisecting AB and DE. Then, since the angles PHC and PVC are right angles (17, *Cor.*), by the last, the points V and H must lie in the circumference of a circle of which CP is the diameter. Hence the required locus is the circumference of a circle having for diameter the line joining the given point and the centre of the given circle.

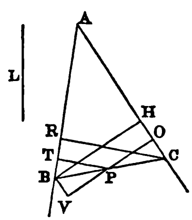
Fig. 93.



4. *To find the locus of a point the sum of whose distances from two intersecting fixed lines is equal to a given straight line.*

If L (Fig. 94) be the given straight line and AB, AC, the two intersecting fixed lines, then, as is evident, some point in AB, or in AB produced, will be at a distance from AC equal to L. Let B be this point, then, drawing BH perpendicular to AC, BH = L. So also some point, as C, in AC or AC produced, is at a distance from AB equal to L; therefore, drawing CR perpendicular to AB, CR = L = BH. Now B and C must be points in the locus since they satisfy the conditions. Draw BC, and because the two triangles BHC, CRB, are equal (59, *Cor.* i.), the triangle ABC is isosceles. Also BC is the required locus; for let P be any point in BC, draw the perpendiculars PO, PT, and produce OP to V, so that OV = HB; then BV is parallel to HO (67), and the right-angled triangles BTP, BVP, are equal, having the side BP common, and the angle PBV = PCO (34) = TBP. Consequently PV = PT, and PT + PO = OV = BH = L.

Fig. 94.



By assuming a point outside BC and between AB and AC, it may easily be shown that the sum of its distances from AB and AC cannot be equal to L. Hence the partial locus is BC, the base of an isosceles triangle such that perpendiculars from its extremities to the opposite sides are each equal to the given line.

By producing AB and AC both ways four such triangles may be formed, and the bases of all the triangles will be the complete locus.

5. The following loci are easily found :—

Of a point at a given distance from a given straight line.

Of a point equidistant from two given parallel lines.

Of a point at a given distance from the circumference of a given circle.

Of a point which bisects equal chords in a given circle.

Of a point bisecting straight lines drawn from a given point to a given line.

Of a point the difference of whose distances from two given lines is equal to a given straight line.

Of a point passing through the centres of circles whose circumferences intersect in two given points.

Of a point at which tangents to a given circle form a given angle.

A ladder standing upright against a wall is lowered by drawing out the foot of the ladder from the wall; required the locus of its centre.

6. A point which is found in each of two loci evidently must coincide with a point in which the loci intersect. Hence the intersection of loci may frequently be employed to determine the position of a point.

Thus the point in a given straight line which is equidistant from two given points is determined by drawing the locus of a point equidistant from the two given points (i.), and the point in which this locus cuts the other locus, which is the given line, is the required point. If the two loci coincide, any point in the given line is the required point; but, if they are parallel, the solution is impossible.

So also a point in a given straight line and at equal distances from two other straight lines is determined by drawing the locus of a point equidistant from the two other lines (i.), and the point or points of the intersection of this locus with the given straight line is the required point.

PART II.

SECTION I.

ON AREAS.

THE *area* of a figure is the extent of its surface, or the relation between its surface and that of another figure taken as a unit of measurement.

The word "surface" is frequently employed in the sense of area.

103. Two figures which are equal in area are said to be *equivalent*. Hence all equal figures (53) are equivalent; but figures may be equivalent without being equal. A parallelogram and a triangle may have equal areas, but cannot be equal figures.

Any side of a parallelogram being regarded as base, the perpendicular to it from the opposite side is called the *altitude* of the parallelogram; and a perpendicular from the vertex of a triangle to the base is called the *altitude* of the triangle.

A parallelogram is sometimes designated by two letters placed at opposite angles.

THEOREM.

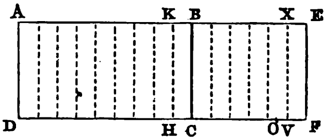
104. *Two rectangles having the same altitude are related to each other as their bases.*

Let ABCD and BEFC (Fig. 95) be two rectangles having the same altitude AD; also let the area of the first be

represented by R, and the area of the second by r; then R : r : : DC : CF.

1°. Let CD and CF be commensurable; and let CH, which is a measure of both, be contained m times in CD and n times in CF. Then, drawing lines parallel to AD through the different points of division made by CH along the line DF, the rectangle BD contains m rectangles each equal to BH, and the rectangle BF contains n such rectangles.

Fig. 95.



Therefore

$$R = BD = m \times BH,$$

$$r = BF = n \times BH;$$

and

$$R : r : : m \times BH : n \times BH : : m : n$$

$$: : m \times CH : n \times CH : : CD : CF.$$

2°. Let CD and CF be incommensurable; and if

$$R : r : : CD : CF.$$

be not a correct proportion, let

$$R : r : : CD : CO,$$

CO being greater or less than CF.

Divide DC into equal parts, each part being less than OF, and continue the division along CF; then one, at least, of the points of division must fall between O and F. Let this point be V; draw VX parallel to AD, and, because CD and CV are commensurable, we have, by the first case, representing the rectangle BV by r',

$$R : r' : : CD : CV;$$

but, by the hypothesis,

$$R : r : : CD : CO,$$

a proportion clearly incompatible with the preceding; for since $r > r'$, CO must be greater than CV.

Therefore CO cannot be less than CF, and in the same way it may be proved that CO cannot be greater than CF. Consequently CO = CF, and

$$R : r :: CD : CF,$$

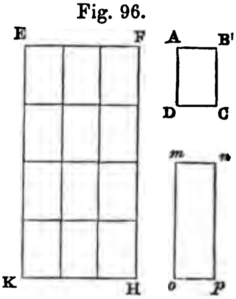
when CD and CF are incommensurable.

THEOREM.

105. *Two rectangles are related to each other as the products of their bases and altitudes respectively.*

Let the areas of the rectangles EH and AC (Fig. 96), be represented by R and r respectively; also let the area of mp, a third rectangle having op = AB, and om = EF, be represented by r'. Then, op being regarded as the base of the rectangle mp,

Fig. 96.



$$R : r' :: EK : op;$$

and, om being regarded as the base of mp,

$$r' : r :: om : AD \text{ (104)}.$$

Therefore, from the two proportions, we get

$$R : r :: EK \times om : op \times AD,$$

or, since om = EF, and op = AB,

$$R : r :: EK \times EF : AD \times AB.$$

THEOREM.

106. *The area of a rectangle is measured by the product of its base and altitude.*

To *measure* or find the area of a figure is to determine how often its surface contains a given unit surface. The unit conventionally adopted is the area of a square constructed on the unit of length. If, therefore, in the preceding theorem we suppose

$$AD = AB = \text{the linear unit},$$

the surface AC is the unit surface, and the proportion

$$R : r :: EK \times EF : AD \times AB$$

becomes

$$R : \text{unit surface} :: EK \times EF : 1.$$

Therefore

$$\frac{R}{\text{unit surface}} = EK \times EF,$$

that is, the product of the numbers which express how often EK and EF respectively contain the unit of length represents how often the rectangle EH, of which EK and EF are the adjacent sides, contains the unit of surface. It is in this sense only that the product of the base and altitude of a rectangle is said to measure, or to be equal to, its area.

Hence the product of any two numbers whatsoever is frequently called their *rectangle*.

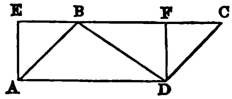
Cor. i. A square being a rectangle in which the adjacent sides are equal, the area of a square is measured by the second power of one of its sides.

From this circumstance the term *square* has come to be synonymous with *second power* in arithmetic and algebra.

Cor. ii. *Any parallelogram is measured by the product of its base and altitude;* for any parallelogram as ABCD (Fig. 97) is equivalent to a rectangle AEFD having the same base

and altitude, since the two right-angled triangles DFC and AEB are equal (59, *Cor.* i., 66), one taken from, the other added to the parallelogram to form the rectangle.

Fig. 97.



Cor. iii. From the last *Cor.* it follows that two parallelograms are equivalent—

1°. When they have equal bases and equal altitudes;

2°. When they have the same or equal bases and are between the same or equidistant parallels (32);

3°. When they have their bases and altitudes reciprocally proportional; because then the product of the base and altitude of one is equal to the product of the base and altitude of the other.

Theorem.

107. *The area of a triangle is measured by half the product of its base and altitude.*

Any triangle as ABD (Fig. 97) can be completed into a parallelogram AC having the same base and altitude as the triangle, but a double area (66, *Cor.* i.). The area of the parallelogram is measured by the product of its base and altitude (106, *Cor.* ii.), therefore the area of the triangle is measured by half the product of its base and altitude.

Cor. i. Two triangles are equivalent in the following cases:—

1°. When they have equal bases and equal altitudes;

2°. When they have the same or equal bases, and are contained by the same or equidistant parallels;

3°. When their bases and altitudes are reciprocally proportional.

The same as for parallelograms.

Cor. ii. The area of a right-angled triangle is measured by half the product of the two small sides.

Cor. iii. A triangle and a parallelogram with the same or equal bases, and having the same or an equal altitude, or

contained by the same or equidistant parallels, have their areas as 1 : 2.

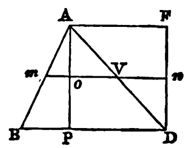
Cor. iv. The area and base of a triangle being given, the locus of its vertex is two lines parallel to the base (32, *Cor.*).

THEOREM.

108. *The area of a trapezium is measured by the product of half the sum of the two parallel sides and the perpendicular between them, which is the altitude of the trapezium.*

The trapezium ABDF (Fig. 98), the sides AF and BD of which are parallel, is divided by the diagonal AD into two triangles having an equal altitude, since a line from D perpendicular to AF, or AF produced, is equal to AP the altitude of the triangle BAD (32). Therefore the area of the trapezium, which is the sum of the areas of the two triangles, is equal to

Fig. 98.



$$\frac{BD}{2} \times AP + \frac{AF}{2} \times AP = \frac{BD + AF}{2} \times AP.$$

If AP be bisected, and a line mn be drawn through the middle point o parallel to BD, this line will bisect AD in V (78), and

$$mV = \frac{BD}{2},\ nV = \frac{AF}{2}\ (78, \textit{Cor. ii}).$$

Therefore

$$\text{area of trapezium} = AP \times mn.$$

Cor. A parallelogram and a trapezium of the same altitude are equivalent when the base of the parallelogram is

equal to a line parallel to, and equidistant from the parallel sides of the trapezium, or to half the sum of the parallel sides.

THEOREM.

109. *The area of a regular polygon is measured by half the product of its perimeter and right radius.*

Any regular polygon as ABDEFG (Fig. 63) is divided by the oblique radii CA, CB, &c., into as many equal triangles as there are sides (57). If, therefore, n be the number of sides, n times the area of any one triangle as ACB will be the area of the polygon; that is,

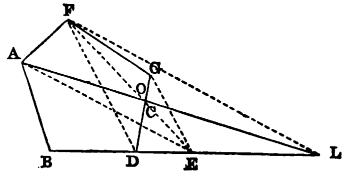
$$\frac{n \times \text{AB} \times \text{CP}}{2} = \text{area of polygon};$$

CP being the altitude of the triangle ACB, or the right radius of the polygon. But $n \times$ AB is the perimeter of the polygon; therefore half the product of the perimeter and right radius measures the surface of any regular polygon.

To obtain the area of an irregular polygon, it may be divided into triangles by drawing diagonals from one of the angles, and the sum of the areas of the triangles will be the area of the polygon; or, any irregular polygon may be reduced to an equivalent triangle as follows:—

From the angle F (Fig. 99) of the irregular polygon ABDGF draw the diagonal FD; and from the angle G the line GE parallel to FD, and meeting the side BD produced. Join FE, and the quadrilateral ABEF is equivalent to the pentagon ABDGF. For the triangles FGD and FDE are equal in area, because they are on the same base FD, and between the parallels FD, GE (107, *Cor.* i.). The common

Fig. 99.



part FCD being subtracted from each, the remainders DCE and FCG are equivalent, the first added to, the second subtracted from the pentagon to form the quadrilateral. Now draw the diagonal AE and its parallel FL to the side BD produced. Join AL, and the triangle BAL is equivalent to the quadrilateral ABEF. For the triangles LAF and FEL are equivalent (107, *Cor.* i.). Subtracting from each the common part FOL, the remainders EOL and AOF are equal in area, one added to, the other subtracted from the quadrilateral to form the triangle ABL, which is, therefore, equivalent to the pentagon ABDGF.

THEOREM.

110. *The area of a circle is measured by half the product of its radius and circumference.*

For (96, *Cor.* ii.) a circle may be regarded as a regular polygon of an indefinite number of sides, in which the right and oblique radii are equal; consequently, half the product of the radius and the perimeter or circumference measures its surface.

If, therefore, r = radius of circle,

$$2\,\pi r = \text{the circumference (102, V.),}$$

and

$$\frac{2\,\pi r \times r}{2} = \pi r^2 = \text{area of circle.}$$

Thus, the area of a circle whose radius is 3 feet $= 3{\cdot}14159 \times 9 = 28{\cdot}27431$ square feet.

The area of a circle whose diameter is one foot is $3{\cdot}14159 \times \frac{1}{4} = 0{\cdot}7854$ square feet, and so on.

Cor. i. *The area of a sector is measured by half the product of its arc and the radius of the circle;* for any sector as BCD

(Fig. 63) is obviously the same part of the entire circle that its arc is of the circumference; that is,

$$\frac{\text{sector}}{\text{circle}} = \frac{\text{arc}}{\text{circumference}},$$

therefore,

$$\text{sector} = \frac{\text{circle} \times \text{arc}}{\text{circumference}} = \frac{r \times \text{circumf.} \times \text{arc}}{2 \times \text{circumf.}} = \frac{r \times \text{arc}}{2}.$$

Cor. ii. The area of a segment is the difference of the areas of the sector having the same arc and the triangle whose base is the chord of the segment, and vertex, the centre of the circle.

111. From what is stated in No. 106, it follows that a twofold meaning, one an arithmetical, the other a geometrical meaning, may be given to each of the expressions

$$a \times b,\ a^2,\ (a+b)^2,\ (a-b)^2,\ (a^2-b^2),\ \&c.,$$

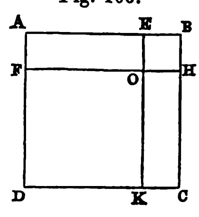
a and b being any two numbers whatsoever.

Thus, if $a = 5$, and $b = 2$,

$$a \times b = 10,$$

or, $a \times b =$ the rectangle AEOF (Fig. 100) in which the side AE contains a units of length and the side EO contains b units of length (106).

Fig. 100.



So also $a^2 = 25$,
or, $a^2 =$ the square FOKD constructed on the line FO which contains a units of length.

And, in general, wherever the second power of a line or the product of two lines occurs, there is always implied a geometrical result corresponding to the arithmetical or algebraic result.

This will be made clearer by the theorems which follow.

THEOREM.

112. *A square constructed on the sum of two given lines is equivalent to the sum of the squares constructed on each of the lines, plus twice the rectangle which they contain.*

Let AB (Fig. 100) be the sum of the lines AE and EB, construct the square AC, and, making BH = BE, draw the lines EK, HF, parallel to BC and BA respectively.

The square AC consists of the squares BO, OD, and of the two equal rectangles OA, OC. The square BO is the square on BE, one of the given lines; the square OD is the square on OF = AE, the other given line; and the two equal rectangles are contained by the sides AE, EO (= EB), and OK (= AE), OH (= EB). Therefore, &c.

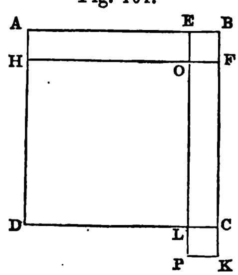
If AE contains a units of length, and BE contains b units, then AB contains $a+b$ units; and (106, *Cor.* i.) the square $AC = (a+b)^2 = a^2 + b^2 + 2\,ab$—the same result as that obtained by geometrical construction.

THEOREM.

113. *A square constructed on the difference of two given lines is equivalent to the sum of the squares on each of the lines, minus twice the rectangle which they contain.*

Let AE (Fig. 101) be the difference of the two given lines AB and EB; on AB construct the square AC, and, taking BF = BE, draw the lines EL, FH parallel to BC and AB respectively. Produce BC to K, so that CK = BF, and complete the rectangle OK.

Fig. 101.



The two rectangles BH and OK are obviously equal, as are also the squares BO and CP. From the figure, the square OD is equivalent to the sum of the squares AC and CP, less

the sum of the two equal rectangles BH and OK, or less twice the rectangle BH.

But OD is the square on HO = AE = AB − EB ;

AC is the square on AB ;

CP is the square on CL = EB ;

and twice the rectangle BH is twice the rectangle contained by the lines AB, BF (= EB). Therefore, &c.

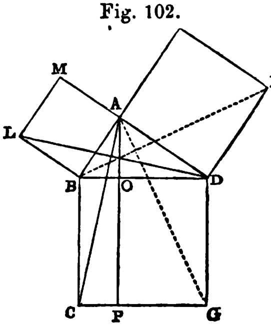
If AB and EB contain respectively a and b units of length, AE contains $a - b$ units of length, and (106, *Cor.* i.) the square on AE $= (a - b)^2 = a^2 + b^2 - 2\,ab$—the same result as above.

THEOREM.

114. *In a right-angled triangle the square on the hypothenuse is equivalent to the sum of the squares on the two small sides.*

Let the triangle BAD (Fig. 102) have a right angle at A. On the sides BA, AD, BD, construct the squares BM, AE, BG respectively. Join A and C, L and D, and draw AP perpendicular to BD, or parallel to BC.

Fig. 102.



The triangle ABC is equivalent to one-half of the rectangle BP (107, *Cor.* iii.), and the triangle LBD is equivalent to one-half of the square BM, DAM being a straight line (10). But the triangles LBD and ABC are equal (54), since AB = LB, BC = BD, and the angle ABC = ABD + a right angle = LBD ; therefore the square BM is equivalent to the rectangle BP.

In a similar way it may be shown that the square AE is equivalent to the rectangle DP.

Consequently the sum of the two rectangles BP and DP,

that is, the square BG, is equivalent to the sum of the two squares BM and AE.

If the lines BD, BA, and AD contain respectively a, b, and c units of length, then (90) $a^2 = b^2 + c^2$, which, interpreted geometrically (111), is the same result as that obtained by the foregoing construction.

In a similar way an extended meaning may be given to the theorems in Nos. 91, 92, 97, 98, 99. . . . .

Cor. If AB = AD, the square on BD is double of the square on AB.

Hence, a square constructed on the diagonal of another square has an area which is twice as great as that of the other square.

Also, calling d and l the diagonal and side of a square respectively, since

$$d^2 = 2l^2,$$

therefore,

$$d = l\sqrt{2}.$$

Hence the diagonal and side of a square are incommensurable, $\sqrt{2}$ being irrational.

If $l = 1$, $d = \sqrt{2}$, that is, the square root of 2, which can only be expressed *approximately* by numbers, can be represented *exactly* by a line d in terms of a unit length l.

The hypothenuse of a right-angled triangle having d and a line equal to l for the other sides, will represent $\sqrt{3}$ in terms of the same unit. And, in a similar way, lines may be drawn which will represent exactly, in terms of the unit l, the square roots of all integral numbers.

SECTION II.

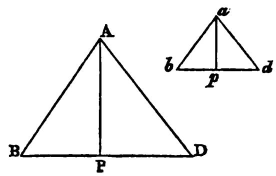
RELATIONS OF AREAS.

THEOREM.

115. *The areas of similar triangles are related to each other as the squares of their homologous dimensions.*

Let S and s be the areas of the two similar triangles BAD, bad (Fig. 103). From the vertices of the equal angles A and a draw the perpendiculars AP, ap; and, since $B = b$, the two triangles BAP, bap are also similar. Hence (86)

Fig. 103.



$$AP : ap :: AB : ab :: BD : bd.$$

But since $$S = \frac{BD \times AP}{2},$$

and $$s = \frac{bd \times ap}{2} \text{ (107)},$$

$$S : s :: BD \times AP : bd \times ap.$$

Comparing this with the first proportion, the middle ratio being neglected, we get

$$S : s :: BD^2 : bd^2 :: AB^2 : ab^2 :: AP^2 : ap^2 \ldots .$$

THEOREM.

116. *The areas of two triangles which have two angles equal are related to each other as the products of the sides containing the equal angles.*

Let S and s be the areas of the two triangles BAD, bad (Fig. 103) in which $B = b$. From either of the other

angles, in each triangle, draw the perpendiculars AP, *ap*. Then

$$S : s :: BD \times AP : bd \times ap;$$

and, since the triangles BAP, *bap* are similar,

$$AP : ap :: BA : ba.$$

Therefore

$$S : s :: BD \times BA : bd \times ba.$$

Cor. The areas of equiangular parallelograms are related to each other as the products of two contiguous sides; for, by drawing diagonals, equiangular parallelograms can be divided into pairs of triangles having two angles equal, the sides containing the equal angles being also the sides of the parallelograms, and the areas of the triangles the halves of the areas of the parallelograms respectively.

Theorem.

117. *The areas of similar polygons are related to each other as the squares of their homologous dimensions.*

The two similar polygons ABDEF, *abdef* (Fig. 82) are divided by the diagonals AD, AE, *ad*, *ae*, into the same number of triangles similar two and two (84). Let S and *s* be the areas of the two polygons; also let the areas of the triangles ABD, DAE, EAF, be represented by T, T_1, T_2, and the areas of *abd*, *dae*, *eaf*, by t, t_1, t_2, respectively. Then

$$T : t :: AB^2 : ab^2$$

$$T_1 : t_1 :: DE^2 : de^2$$

$$T_2 : t_2 :: AF^2 : af^2 \ (115).$$

But

$$AB^2 : ab^2 :: BD^2 : bd^2 :: DE^2 : de^2 \ \ldots\ldots \ (94, \textit{Def.})$$

therefore,

$$T : t : : T_1 : t_1 : : T_2 : t_2,$$

and

$$T + T_1 + T_2 : t + t_1 + t_2 : : T : t : : AB^2 : ab^2 \; . \; . \; . \; . \; ;$$

that is,

$$S : s : : AB^2 : ab^2 : : BD^2 : bd^2, \text{ \&c.}$$

Cor. i. *The areas of two regular polygons of the same number of sides are related to each other as the squares of their right and oblique radii* (96).

Cor. ii. If R and r be the radii of two circles whose areas are respectively C and c, then, since $C = \pi R^2$, $c = \pi r^2$ (110),

$$C : c : : R^2 : r^2 : : D^2 : d^2 \; . \; . \; . \; .$$

118. In general, if S and s are the areas of any two figures, and A and B, a and b, the dimensions which measure them respectively, then

$$S = A \times B, \; s = a \times b,$$

and

$$S : s : : A \times B : a \times b.$$

If $A = a$,

$$S : s : : B : b;$$

if $B = b$,

$$S : s : : A : a;$$

and if

$$A : a : : b : B,$$

then, $A \times B = a \times b$, and $S = s$. Hence

Two triangles, or two parallelograms, having the same altitude are related to each other as their bases.

Also a triangle and a parallelogram of the same altitude are equal in area when the base of the parallelogram is half the base of the triangle.

Theorem.

119. *The three sides of a right-angled triangle being homologous dimensions of three similar figures, the area of any figure whatsoever on the hypothenuse is equivalent to the sum of the areas of the two similar figures on the small sides.*

Let S, s, s_1, be the areas of the figures constructed on the hypothenuse and the small sides, and A, a, b, the units of length which the hypothenuse and sides respectively contain.

Then as the three sides of the right-angled triangle are *homologous* dimensions of similar figures,

$$S : s : A^2 : a^2 \text{ (117)},$$

and

$$s : s_1 : : a^2 : b^2.$$

Therefore

$$s : s + s_1 : : a^2 : a^2 + b^2;$$

and, comparing this with the first proportion,

$$S : s + s_1 : : A^2 : a^2 + b^2.$$

But

$$A^2 = a^2 + b^2 \text{ (90)},$$

consequently,

$$S = s + s_1.$$

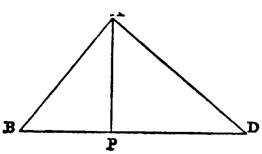
Theorem.

120. *In a right-angled triangle if a perpendicular is drawn from the vertex of the right angle to the hypothenuse, the square or any other figure on the hypothenuse is to the square or other similar figure on either of the small sides as the hypothenuse is to the segment contiguous to the small side.*

Let S and s be the areas of similar figures in which BD and BA (Fig. 104) are homologous dimensions. Then

$$S : s : : BD^2 : BA^2 \text{ (117)}.$$

Fig. 104.



But, AP being a perpendicular from the right angle to the hypothenuse,

$$BD : BA :: BA : BP \text{ (89, } Cor.\text{)}.$$

Therefore

$$BD : BP :: BD^2 : BA^2;$$

and

$$S : s :: BD^2 : BA^2 :: BD : BP.$$

Problems.

121. I. *To construct a circle equivalent to the sum of any number of given circles.*

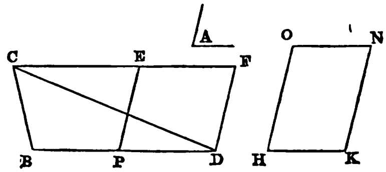
Draw AB and AD (Fig. 104) at right angles to each other and equal respectively to the diameters of two of the given circles. The hypothenuse BD is the diameter of a circle which is equivalent to the sum of the other two (119). Next draw a line perpendicular to BD at either extremity and equal to the diameter of a third given circle. The line which completes the triangle is the hypothenuse, and the diameter of a circle equivalent to the three given circles (119).

By continuing this process, the hypothenuse of a right-angled triangle may be found which will be also the diameter of a circle equivalent to any number of given circles.

In the same way a figure may be constructed equivalent to the sum of any number of similar figures.

II. *On a given line to construct a parallelogram equivalent to a given triangle and having an angle equal to a given angle.*

Fig. 105.



Let HK (Fig. 105) be the given line, BCD the given triangle, and A the given angle.

Bisect BD in P, draw through C a line parallel to BD, and through P a line PE making an

THEOREM.

106. *The area of a rectangle is measured by the product of its base and altitude.*

To *measure* or find the area of a figure is to determine how often its surface contains a given unit surface. The unit conventionally adopted is the area of a square constructed on the unit of length. If, therefore, in the preceding theorem we suppose

$$AD = AB = \text{the linear unit},$$

the surface AC is the unit surface, and the proportion

$$R : r : : EK \times EF : AD \times AB$$

becomes

$$R : \text{unit surface} : : EK \times EF : 1.$$

Therefore

$$\frac{R}{\text{unit surface}} = EK \times EF,$$

that is, the product of the numbers which express how often EK and EF respectively contain the unit of length represents how often the rectangle EH, of which EK and EF are the adjacent sides, contains the unit of surface. It is in this sense only that the product of the base and altitude of a rectangle is said to measure, or to be equal to, its area.

Hence the product of any two numbers whatsoever is frequently called their *rectangle.*

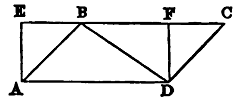
Cor. i. A square being a rectangle in which the adjacent sides are equal, the area of a square is measured by the second power of one of its sides.

From this circumstance the term *square* has come to be synonymous with *second power* in arithmetic and algebra.

Cor. ii. *Any parallelogram is measured by the product of its base and altitude;* for any parallelogram as ABCD (Fig. 97) is equivalent to a rectangle AEFD having the same base

and altitude, since the two right-angled triangles DFC and AEB are equal (59, *Cor.* i., 66), one taken from, the other added to the parallelogram to form the rectangle.

Fig. 97.



Cor. iii. From the last *Cor.* it follows that two parallelograms are equivalent—

1°. When they have equal bases and equal altitudes;

2°. When they have the same or equal bases and are between the same or equidistant parallels (32);

3°. When they have their bases and altitudes reciprocally proportional; because then the product of the base and altitude of one is equal to the product of the base and altitude of the other.

Theorem.

107. *The area of a triangle is measured by half the product of its base and altitude.*

Any triangle as ABD (Fig. 97) can be completed into a parallelogram AC having the same base and altitude as the triangle, but a double area (66, *Cor.* i.). The area of the parallelogram is measured by the product of its base and altitude (106, *Cor.* ii.), therefore the area of the triangle is measured by half the product of its base and altitude.

Cor. i. Two triangles are equivalent in the following cases:—

1°. When they have equal bases and equal altitudes;

2°. When they have the same or equal bases, and are contained by the same or equidistant parallels;

3°. When their bases and altitudes are reciprocally proportional.

The same as for parallelograms.

Cor. ii. The area of a right-angled triangle is measured by half the product of the two small sides.

Cor. iii. A triangle and a parallelogram with the same or equal bases, and having the same or an equal altitude, or

contained by the same or equidistant parallels, have their areas as 1 : 2.

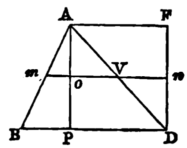
Cor. iv. The area and base of a triangle being given, the locus of its vertex is two lines parallel to the base (32, *Cor.*).

THEOREM.

108. *The area of a trapezium is measured by the product of half the sum of the two parallel sides and the perpendicular between them, which is the altitude of the trapezium.*

The trapezium ABDF (Fig. 98), the sides AF and BD of which are parallel, is divided by the diagonal AD into two triangles having an equal altitude, since a line from D perpendicular to AF, or AF produced, is equal to AP the altitude of the triangle BAD (32). Therefore the area of the trapezium, which is the sum of the areas of the two triangles, is equal to

Fig. 98.



$$\frac{BD}{2} \times AP + \frac{AF}{2} \times AP = \frac{BD + AF}{2} \times AP.$$

If AP be bisected, and a line mn be drawn through the middle point o parallel to BD, this line will bisect AD in V (78), and

$$mV = \frac{BD}{2},\ nV = \frac{AF}{2}\ (78,\ \textit{Cor.}\ \text{ii}).$$

Therefore

$$\text{area of trapezium} = AP \times mn.$$

Cor. A parallelogram and a trapezium of the same altitude are equivalent when the base of the parallelogram is

equal to a line parallel to, and equidistant from the parallel sides of the trapezium, or to half the sum of the parallel sides.

THEOREM.

109. *The area of a regular polygon is measured by half the product of its perimeter and right radius.*

Any regular polygon as ABDEFG (Fig. 63) is divided by the oblique radii CA, CB, &c., into as many equal triangles as there are sides (57). If, therefore, n be the number of sides, n times the area of any one triangle as ACB will be the area of the polygon; that is,

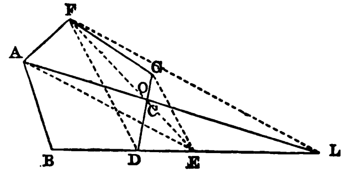
$$\frac{n \times \mathrm{AB} \times \mathrm{CP}}{2} = \text{area of polygon};$$

CP being the altitude of the triangle ACB, or the right radius of the polygon. But $n \times \mathrm{AB}$ is the perimeter of the polygon; therefore half the product of the perimeter and right radius measures the surface of any regular polygon.

To obtain the area of an irregular polygon, it may be divided into triangles by drawing diagonals from one of the angles, and the sum of the areas of the triangles will be the area of the polygon; or, any irregular polygon may be reduced to an equivalent triangle as follows:—

From the angle F (Fig. 99) of the irregular polygon ABDGF draw the diagonal FD; and from the angle G the line GE parallel to FD, and meeting the side BD produced. Join FE, and the quadrilateral ABEF is equivalent to the pentagon ABDGF. For the triangles FGD and FDE are equal in area, because they are on the same base FD, and between the parallels FD, GE (107, *Cor.* i.). The common

Fig. 99.



part FCD being subtracted from each, the remainders DCE and FCG are equivalent, the first added to, the second subtracted from the pentagon to form the quadrilateral. Now draw the diagonal AE and its parallel FL to the side BD produced. Join AL, and the triangle BAL is equivalent to the quadrilateral ABEF. For the triangles LAF and FEL are equivalent (107, *Cor.* i.). Subtracting from each the common part FOL, the remainders EOL and AOF are equal in area, one added to, the other subtracted from the quadrilateral to form the triangle ABL, which is, therefore, equivalent to the pentagon ABDGF.

THEOREM.

110. *The area of a circle is measured by half the product of its radius and circumference.*

For (96, *Cor.* ii.) a circle may be regarded as a regular polygon of an indefinite number of sides, in which the right and oblique radii are equal; consequently, half the product of the radius and the perimeter or circumference measures its surface.

If, therefore, r = radius of circle,

$$2\,\pi r = \text{the circumference (102, V.),}$$

and

$$\frac{2\,\pi r \times r}{2} = \pi r^2 = \text{area of circle.}$$

Thus, the area of a circle whose radius is 3 feet = $3{\cdot}14159 \times 9 = 28{\cdot}27431$ square feet.

The area of a circle whose diameter is one foot is $3{\cdot}14159 \times \frac{1}{4} = 0{\cdot}7854$ square feet, and so on.

Cor. i. *The area of a sector is measured by half the product of its arc and the radius of the circle;* for any sector as BCD

(Fig. 63) is obviously the same part of the entire circle that its arc is of the circumference; that is,

$$\frac{\text{sector}}{\text{circle}} = \frac{\text{arc}}{\text{circumference}},$$

therefore,

$$\text{sector} = \frac{\text{circle} \times \text{arc}}{\text{circumference}} = \frac{r \times \text{circumf.} \times \text{arc}}{2 \times \text{circumf.}} = \frac{r \times \text{arc}}{2}.$$

Cor. ii. The area of a segment is the difference of the areas of the sector having the same arc and the triangle whose base is the chord of the segment, and vertex, the centre of the circle.

111. From what is stated in No. 106, it follows that a twofold meaning, one an arithmetical, the other a geometrical meaning, may be given to each of the expressions

$$a \times b,\ a^2,\ (a+b)^2,\ (a-b)^2,\ (a^2-b^2),\ \&c.,$$

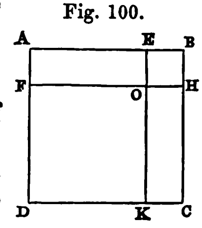
a and b being any two numbers whatsoever.

Thus, if $a = 5$, and $b = 2$,

$$a \times b = 10,$$

or, $a \times b$ = the rectangle AEOF (Fig. 100) in which the side AE contains a units of length and the side EO contains b units of length (106).

Fig. 100.



So also $a^2 = 25$,
or, a^2 = the square FOKD constructed on the line FO which contains a units of length.

And, in general, wherever the second power of a line or the product of two lines occurs, there is always implied a geometrical result corresponding to the arithmetical or algebraic result.

This will be made clearer by the theorems which follow.

THEOREM.

112. *A square constructed on the sum of two given lines is equivalent to the sum of the squares constructed on each of the lines, plus twice the rectangle which they contain.*

Let AB (Fig. 100) be the sum of the lines AE and EB, construct the square AC, and, making BH = BE, draw the lines EK, HF, parallel to BC and BA respectively.

The square AC consists of the squares BO, OD, and of the two equal rectangles OA, OC. The square BO is the square on BE, one of the given lines; the square OD is the square on OF = AE, the other given line; and the two equal rectangles are contained by the sides AE, EO (= EB), and OK (= AE), OH (= EB). Therefore, &c.

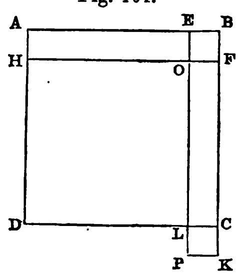
If AE contains a units of length, and BE contains b units, then AB contains $a+b$ units; and (106, *Cor.* i.) the square AC $= (a+b)^2 = a^2 + b^2 + 2\,ab$—the same result as that obtained by geometrical construction.

THEOREM.

113. *A square constructed on the difference of two given lines is equivalent to the sum of the squares on each of the lines, minus twice the rectangle which they contain.*

Let AE (Fig. 101) be the difference of the two given lines AB and EB; on AB construct the square AC, and, taking BF = BE, draw the lines EL, FH parallel to BC and AB respectively. Produce BC to K, so that CK = BF, and complete the rectangle OK.

Fig. 101.



The two rectangles BH and OK are obviously equal, as are also the squares BO and CP. From the figure, the square OD is equivalent to the sum of the squares AC and CP, less

the sum of the two equal rectangles BH and OK, or less twice the rectangle BH.

But OD is the square on HO = AE = AB − EB;

AC is the square on AB;

CP is the square on CL = EB;

and twice the rectangle BH is twice the rectangle contained by the lines AB, BF (= EB). Therefore, &c.

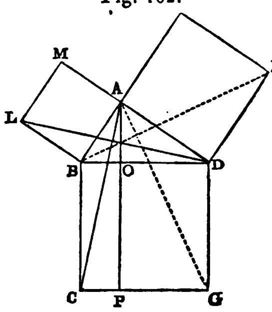
If AB and EB contain respectively a and b units of length, AE contains $a - b$ units of length, and (106, *Cor.* i.) the square on AE $= (a - b)^2 = a^2 + b^2 - 2ab$—the same result as above.

THEOREM.

114. *In a right-angled triangle the square on the hypothenuse is equivalent to the sum of the squares on the two small sides.*

Let the triangle BAD (Fig. 102) have a right angle at A. On the sides BA, AD, BD, construct the squares BM, AE, BG respectively. Join A and C, L and D, and draw AP perpendicular to BD, or parallel to BC.

Fig. 102.



The triangle ABC is equivalent to one-half of the rectangle BP (107, *Cor.* iii.), and the triangle LBD is equivalent to one-half of the square BM, DAM being a straight line (10). But the triangles LBD and ABC are equal (54), since AB = LB, BC = BD, and the angle ABC = ABD + a right angle = LBD; therefore the square BM is equivalent to the rectangle BP.

In a similar way it may be shown that the square AE is equivalent to the rectangle DP.

Consequently the sum of the two rectangles BP and DP,

that is, the square BG, is equivalent to the sum of the two squares BM and AE.

If the lines BD, BA, and AD contain respectively a, b, and c units of length, then (90) $a^2 = b^2 + c^2$, which, interpreted geometrically (111), is the same result as that obtained by the foregoing construction.

In a similar way an extended meaning may be given to the theorems in Nos. 91, 92, 97, 98, 99. . . . .

Cor. If AB = AD, the square on BD is double of the square on AB.

Hence, a square constructed on the diagonal of another square has an area which is twice as great as that of the other square.

Also, calling d and l the diagonal and side of a square respectively, since

$$d^2 = 2l^2,$$

therefore,

$$d = l\sqrt{2}.$$

Hence the diagonal and side of a square are incommensurable, $\sqrt{2}$ being irrational.

If $l = 1$, $d = \sqrt{2}$, that is, the square root of 2, which can only be expressed *approximately* by numbers, can be represented *exactly* by a line d in terms of a unit length l.

The hypothenuse of a right-angled triangle having d and a line equal to l for the other sides, will represent $\sqrt{3}$ in terms of the same unit. And, in a similar way, lines may be drawn which will represent exactly, in terms of the unit l, the square roots of all integral numbers.

SECTION II.

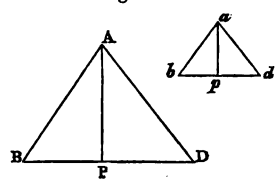
RELATIONS OF AREAS.

THEOREM.

115. *The areas of similar triangles are related to each other as the squares of their homologous dimensions.*

Let S and s be the areas of the two similar triangles BAD, bad (Fig. 103). From the vertices of the equal angles A and a draw the perpendiculars AP, ap; and, since $B = b$, the two triangles BAP, bap are also similar. Hence (86)

Fig. 103.



$$AP : ap :: AB : ab :: BD : bd.$$

But since $$S = \frac{BD \times AP}{2},$$

and $$s = \frac{bd \times ap}{2} \ (107),$$

$$S : s :: BD \times AP : bd \times ap.$$

Comparing this with the first proportion, the middle ratio being neglected, we get

$$S : s :: BD^2 : bd^2 :: AB^2 : ab^2 :: AP^2 : ap^2 \ \ldots .$$

THEOREM.

116. *The areas of two triangles which have two angles equal are related to each other as the products of the sides containing the equal angles.*

Let S and s be the areas of the two triangles BAD, bad (Fig. 103) in which $B = b$. From either of the other

angles, in each triangle, draw the perpendiculars AP, *ap*. Then

$$S : s :: BD \times AP : bd \times ap;$$

and, since the triangles BAP, *bap* are similar,

$$AP : ap :: BA : ba.$$

Therefore

$$S : s :: BD \times BA : bd \times ba.$$

Cor. The areas of equiangular parallelograms are related to each other as the products of two contiguous sides; for, by drawing diagonals, equiangular parallelograms can be divided into pairs of triangles having two angles equal, the sides containing the equal angles being also the sides of the parallelograms, and the areas of the triangles the halves of the areas of the parallelograms respectively.

Theorem.

117. *The areas of similar polygons are related to each other as the squares of their homologous dimensions.*

The two similar polygons ABDEF, *abdef* (Fig. 82) are divided by the diagonals AD, AE, *ad*, *ae*, into the same number of triangles similar two and two (84). Let S and *s* be the areas of the two polygons; also let the areas of the triangles ABD, DAE, EAF, be represented by T, T_1, T_2, and the areas of *abd*, *dae*, *eaf*, by t, t_1, t_2, respectively. Then

$$T : t :: AB^2 : ab^2$$

$$T_1 : t_1 :: DE^2 : de^2$$

$$T_2 : t_2 :: AF^2 : af^2 \text{ (115).}$$

But

$$AB^2 : ab^2 :: BD^2 : bd^2 :: DE^2 : de^2 \ldots\ldots \text{(94, \textit{Def.})}$$

therefore,

$$T : t :: T_1 : t_1 :: T_2 : t_2,$$

and

$$T + T_1 + T_2 : t + t_1 + t_2 :: T : t :: AB^2 : ab^2 \ldots ;$$

that is,

$$S : s :: AB^2 : ab^2 :: BD^2 : bd^2, \text{ \&c.}$$

Cor. i. *The areas of two regular polygons of the same number of sides are related to each other as the squares of their right and oblique radii* (96).

Cor. ii. If R and r be the radii of two circles whose areas are respectively C and c, then, since $C = \pi R^2$, $c = \pi r^2$ (110),

$$C : c :: R^2 : r^2 :: D^2 : d^2 \ldots$$

118. In general, if S and s are the areas of any two figures, and A and B, a and b, the dimensions which measure them respectively, then

$$S = A \times B, \; s = a \times b,$$

and

$$S : s :: A \times B : a \times b.$$

If $A = a$,

$$S : s :: B : b;$$

if $B = b$,

$$S : s :: A : a;$$

and if

$$A : a :: b : B,$$

then, $A \times B = a \times b$, and $S = s$. Hence

Two triangles, or two parallelograms, having the same altitude are related to each other as their bases.

Also a triangle and a parallelogram of the same altitude are equal in area when the base of the parallelogram is half the base of the triangle.

THEOREM.

119. *The three sides of a right-angled triangle being homologous dimensions of three similar figures, the area of any figure whatsoever on the hypothenuse is equivalent to the sum of the areas of the two similar figures on the small sides.*

Let S, s, s_1, be the areas of the figures constructed on the hypothenuse and the small sides, and A, a, b, the units of length which the hypothenuse and sides respectively contain.

Then as the three sides of the right-angled triangle are *homologous* dimensions of similar figures,

$$S : s : A^2 : a^2 \text{ (117)},$$

and

$$s : s_1 : : a^2 : b^2.$$

Therefore

$$s : s + s_1 : : a^2 : a^2 + b^2;$$

and, comparing this with the first proportion,

$$S : s + s_1 : : A^2 : a^2 + b^2.$$

But

$$A^2 = a^2 + b^2 \text{ (90)},$$

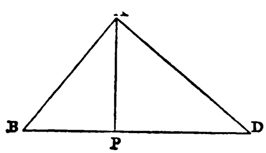
consequently,

$$S = s + s_1.$$

THEOREM.

120. *In a right-angled triangle if a perpendicular is drawn from the vertex of the right angle to the hypothenuse, the square or any other figure on the hypothenuse is to the square or other similar figure on either of the small sides as the hypothenuse is to the segment contiguous to the small side.*

Fig. 104.



Let S and s be the areas of similar figures in which BD and BA (Fig. 104) are homologous dimensions. Then

$$S : s : : BD^2 : BA^2 \text{ (117)}.$$

But, AP being a perpendicular from the right angle to the hypothenuse,

$$BD : BA :: BA : BP \text{ (89, } Cor.\text{)}.$$

Therefore

$$BD : BP :: BD^2 : BA^2;$$

and

$$S : s :: BD^2 : BA^2 :: BD : BP.$$

Problems.

121. I. *To construct a circle equivalent to the sum of any number of given circles.*

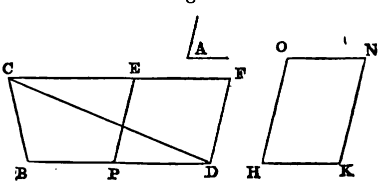
Draw AB and AD (Fig. 104) at right angles to each other and equal respectively to the diameters of two of the given circles. The hypothenuse BD is the diameter of a circle which is equivalent to the sum of the other two (119). Next draw a line perpendicular to BD at either extremity and equal to the diameter of a third given circle. The line which completes the triangle is the hypothenuse, and the diameter of a circle equivalent to the three given circles (119).

By continuing this process, the hypothenuse of a right-angled triangle may be found which will be also the diameter of a circle equivalent to any number of given circles.

In the same way a figure may be constructed equivalent to the sum of any number of similar figures.

II. *On a given line to construct a parallelogram equivalent to a given triangle and having an angle equal to a given angle.*

Fig. 105.



Let HK (Fig. 105) be the given line, BCD the given triangle, and A the given angle.

Bisect BD in P, draw through C a line parallel to BD, and through P a line PE making an

angle DPE = A. Complete the parallelogram DE, and it is equivalent to the triangle BCD (118).

At the point H make an angle KHO = A, take

$$HK : PD :: PE : x = HO \text{ (79, I.)},$$

or,

$$HK \times HO = PD \times PE.$$

Complete the parallelogram KO, and it is the parallelogram required.

For KO is constructed on the given line HK, has an angle KHO = A, and is equivalent to the triangle BCD, since it is equivalent to the parallelogram DE (117, *Cor.*).

As a polygon may be divided into triangles, a parallelogram equivalent to any given polygon may be constructed on any given line.

III. *To find the quadrature of a given figure.*

To square or find the quadrature of a given figure is to find the side of a square which is equivalent to the figure.

The side of a square equivalent to a given triangle is determined by finding a mean proportional (102, I.) between the altitude and half the base (107).

A mean proportional between the altitude and base of a parallelogram is the side of an equivalent square (106, *Cor.* ii.).

By reducing any polygon to an equivalent triangle (109), or parallellogram (121, II.), the side of a square equivalent to the polygon may be found.

Since the area of a circle is approximately expressed by half the product of its circumference and radius (110), the side of a square approximately equal in area to a given circle is a mean proportional between the radius and half the circumference. Or, calling the area of the given circle C, and the side of an equivalent square x,

$$C = \pi r^2 \text{ (110)},$$

and

$$x = r\sqrt{\pi}.$$

For want of an exact value of π (102, V.), the side of a square equivalent to a given circle can only be determined by approximation.

Exercises.

1. A parallelogram is divided by its diagonals into four equivalent triangles.

2. The two figures into which a parallelogram is divided by any straight line which passes through the point of intersection of the diagonals, are equivalent.

3. Equivalent triangles or parallelograms on the same base and on the same side of it are between the same parallels.

4. If through any point in the diagonal of a parallelogram lines are drawn parallel to the sides, the two parallelograms through which the diagonal does not pass are equivalent.

5. The area of a triangle is equal to half the product of its perimeter and the radius of the inscribed circle.

6. Two sides of a triangle being given, the area is greatest when the angle which they form is a right angle.

7. Of all equivalent triangles on the same base the isosceles has the best perimeter.

8. A straight line which bisects the two parallel sides of a trapezium bisects also its surface.

9. If lines are drawn from the middle point of one of the non-parallel sides of a trapezium to the extremities of the opposite side, the area of the triangle which they form is half the area of the trapezium.

10. Two triangles are equivalent when they have an equal angle each, and the sides about the equal angles are respectively proportional.

11. To divide a straight line into two parts so that the rectangle which they contain shall be equivalent to a given rectangle.

12. The area of the parallelogram formed by joining the middle points of the sides of a quadrilateral is half the area of the quadrilateral.

13. How many square feet in a sector whose arc is 60° and radius 12 feet?

14. The difference of the squares constructed on two given lines is equivalent to the rectangle contained by their sum and difference.

15. Prove that the areas of inscribed and circumscribed equilateral triangles are as 1 : 4; of inscribed and circumscribed squares as 1 : 2; of inscribed and circumscribed regular hexagons as 3 : 4.

16. The sides of two equilateral triangles being given, to find the side of a third which is equivalent to the difference of the other two.

17. When a straight line is divided into two unequal segments, the sum of the squares on the segments is equivalent to twice the sum of the squares on half the line and that part of it which lies between the middle point and the point of section.

18. A straight line being divided into two unequal segments, the rectangle contained by the segments is equivalent to the difference of the squares constructed on half the line and the part of it intercepted between the point of section and the middle point.

19. Of all rectangles having a given perimeter the square has the greatest area.

20. Divide a line into two such parts that the square on one of the parts shall be half the square on the whole line.

21. Interpret and represent geometrically each of the following expressions, $a, b, c, d, \ldots$ being the units of length in given lines:—

(1). If $a = m + n + p \ldots$, $a \times b = bm + bn + bp \ldots$,

(2). If $a = m + n$, $a^2 = am + an$.

(3). If $a = m + n$, $an = mn + n^2$.

(4). If $a = m + n$, $a^2 + m^2 = 2am + n^2$.

(5). If $a = m + n + p$, $a^2 = m^2 + n^2 + p^2 + 2mn + 2mp + 2np$.

PART III.

SECTION I.

ON PLANES.

IN the preceding parts of this treatise we have supposed all the lines to be drawn in one plane; we shall now consider lines which lie in different planes, and the intersections of two or more planes with each other.

POSTULATE.

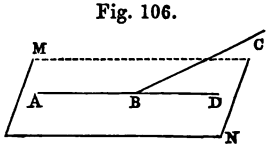
A plane may be indefinitely produced in the directions of its length and breadth.

THEOREM.

122. *A straight line is entirely in a plane in which it has any two of its points.*

If possible let the straight line ABC (Fig. 106) have the points A and B in the plane MN, and the part BC outside it; and let another straight line AD be applied to the plane in the direction ABD. Then (*Def.* VI., Part I.) every point of the line AD is in the plane MN. Consequently the straight lines AD and ABC having the points A and B in common must form

Fig. 106.



one line (*Def.* V., 2, Part I.); that is, ABC must fall on ABD and lie in the plane MN.

Cor. i. *The intersection of two planes is a straight line.*

1°. It is a *line*, for it has length, but neither breadth nor thickness, as the planes have no thickness (*Def.* II., P. I.).

2°. It is a straight line for any two points whatsoever being assumed in the line of intersection, the straight line which joins them must lie in each of the planes, and must, therefore, be the line of intersection, since only one straight line can be drawn between two given points.

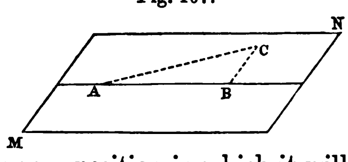
Cor. ii. Two planes cannot intersect in more than one straight line. For if two planes intersect in *two* straight lines, then a line drawn from any point in one of the lines of intersection to any point in the other is also a line of intersection, as having two of its points in each of the planes. But an indefinite number of such lines may be drawn; therefore the planes must coincide.

Theorem.

123. *Through three points which are not in the same straight line, one, and only one, plane can be drawn.*

Let A, B, and C (Fig. 107) be the three points, and through any two of them, as A and B, let a straight line AB, and also a plain MN, be drawn. If now the plane MN be conceived to revolve round AB as an axis, evidently there is one, and only one, position in which it will contain the point C. Hence, as A and B are in the plane in every position which it can assume, one, and only one, plane can pass through the points A, B, and C.

Fig. 107.



Again, if two planes pass through the points A, B, and C, then, drawing AC and BC, each of the lines AB, AC, and BC is in each of the two planes (122); therefore two planes

can intersect in more than one straight line, which is impossible (122, *Cor.* ii.).

Cor. i. *Two straight lines which meet each other have one, and only one, plane in common;* for if AB and AC (Fig. 107) be the two straight lines, then through their meeting point A, and two points, one in each of the lines, as B and C, one plane, and only one plane, can be drawn; also, in this plane AB and AC are situated (122).

Cor. ii. The three sides of a triangle are in one, and only one, plane; for the vertices of the angles are three points not in the same straight line.

Cor. iii. Two parallel lines have only one plane in common; for two planes cannot intersect in one of the parallel lines and pass through a point in the other.

Cor. iv. Through the same point in space only one line can pass so as to be parallel to a given line.

THE RELATIVE POSITION OF LINES AND PLANES.

124. *Def.* A line is said to be perpendicular to a plane when it makes right angles with all the lines in the plane that pass through the point at which it meets the plane.

That a perpendicular to a plane, in the sense of the definition, is possible, will appear from the following

Theorem.

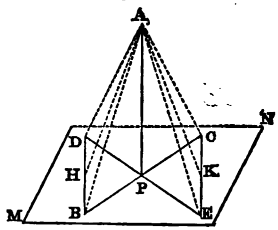
125. *A line is perpendicular to a plane when it makes right angles with any two lines in the plane which pass through the meeting point.*

If AP (Fig. 108) is perpendicular to the lines BC and DE, it is also perpendicular to the plane MN in which these two lines are situate.

Take PB = PC, PD = PE; draw DB, CE, and, from any point H in DB, the line HK through the point P.

The two triangles DPB and CPE are equal (54), therefore DB = CE. The two triangles HPB and KPC are also equal (55), therefore PH = PK, and HB = KC. From the equality of the right-angled triangles APE and APD, APB and APC (54), we get AE = AD, AB = AC. Therefore the two triangles CAE and BAD are equal (57), and the angle ACE = ABD. Consequently the triangles ACK and ABH are equal (54) and AK = AH. Hence each of the points A and P being equidistant from H and K, AP is perpendicular to HK (17), which is *any* line in the plane MN passing through the point P. Consequently AP is perpendicular to the plane MN (124).

Fig. 108.



Cor. i. From the same point outside a plane only one perpendicular can be drawn to the plane; for, if possible, let AP and AD be both perpendicular to the plane MN; then, drawing PD, AP and AD are both perpendicular to PD (124), which is absurd (21).

Cor. ii. A perpendicular is the shortest line that can be drawn from a given point to a plane; for AP is less than any line such as AE drawn from the point A to the plane MN (48).

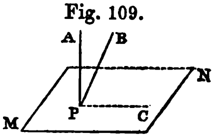
Hence, *a perpendicular is the proper measure of the distance from a point to a plane.*

THEOREM.

126. *To the same point in a plane only one straight line can be drawn so as to be perpendicular to the plane.*

If possible let AP and BP (Fig. 109) be both perpendicular to the plane MN at the point P, and let PC be the intersection of the plane of these lines with the plane MN. Then AP, BP, and PC are all in the same plane, and AP and BP are both perpendicular to PC, which is absurd (14).

Fig. 109.

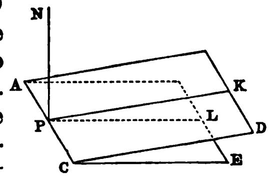


THEOREM.

127. *A line cannot be perpendicular to two planes which it meets at the same point.*

If possible let NP (Fig. 110) be perpendicular to the two planes AD and AE at the point P. Through the point P in the plane AD draw the line PK; and let PL be the intersection of the plane of the two lines PN, PK, with the plane AE. Then PN, PK, PL are all in the same plane; and since NP is perpendicular to the planes AD and AE, it is perpendicular to the lines PK and PL (124), which is absurd (14).

Fig. 110.



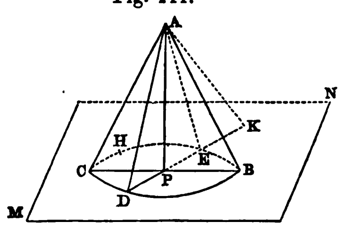
Cor. If a line is perpendicular to three or more lines which it meets at the same point, all these lines are in the same plane (125).

THEOREM.

128. *If a line be perpendicular to a plane, and from the meeting point as centre a circle be described on the plane, any point in the perpendicular will be equidistant from every point in the circumference of the circle.*

Let AP (Fig. 111) be perpendicular to the plane MN, and with P as centre describe a circle BEC on the plane. Then any point A in the perpendicular is equidistant from every point in the circumference of the circle.

Fig. 111.



Join A and the points B, E, C, D . . . . The right-angled triangles APB, APE, APC, ... are all equal, since AP is common, and PB = PE = PC . . . . Therefore AB = AE = AC . . . .

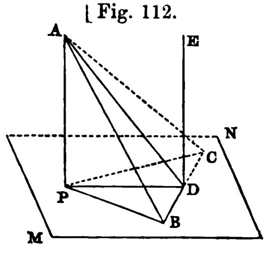
Cor. i. All lines drawn to a plane from the same point in a perpendicular to the plane and equally inclined to the perpendicular are equal, and meet the plane at equal distances from the foot of the perpendicular (55).

Cor. ii. Of two lines drawn to a plane from the same point in a perpendicular to the plane, that is the greater which meets the plane at the greater distance from the foot of the perpendicular. For if AB and AK (Fig. 111) be drawn to the plane MN from the point A in the perpendicular AP so that PK > PB, and with P as centre and PB as radius, a circle BEC be described cutting PK in the point E, then, drawing AE, AB = AE < AK (49).

THEOREM.

129. *If two straight lines be parallel, and one of them be perpendicular to any plane, the other shall be perpendicular to the same plane.*

Fig. 112.



Let AP and ED (Fig. 112) be two parallel lines, one of which, AP, is perpendicular to the plane MN, and let PD be the intersection of their common plane with the plane MN. Draw BC in the plane MN at right angles to PD and make DB = DC. Draw, also, PB, PC, AB, AC.

From the two equal right-angled triangles PDB, PDC, PB = PC; and from the two, APB, APC, AB = AC. Hence the triangles ADB, ADC, are right-angled and equal (57). Therefore, the angles BDA and BDP being both right angles, BD is perpendicular to the plane of the triangle ADP (125), which is the plane of the parallels AP and ED. Consequently the line ED, being perpendicular to the line DB and also to the line DP (31), in the plane MN, is perpendicular to the plane MN (125).

Cor. i. *Two lines perpendicular to the same plane are parallel.*

Let AP and ED (Fig. 112) be both perpendicular to the plane MN, then ED is parallel to AP. For if not, through the point D a line may be drawn parallel to AP (37, I.); but a line through D parallel to AP will be perpendicular to the plane MN, and must coincide with ED, since two perpendiculars cannot be drawn to the same plane at the same point (126).

Cor. ii. Two lines which are parallel to a third are parallel to each other even when the three lines have no one plane in common; for a plane perpendicular to the third line will be perpendicular to each of the others which, therefore, by the last *Cor.*, must be parallel.

130. When a perpendicular is drawn to a plane from a point outside the plane, the foot of the perpendicular is called the *projection* of the point on the plane.

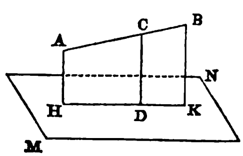
The *projection of a line* on a plane is the line formed by projecting all the points of the line on the plane.

THEOREM.

The projection of a straight line on a plane is a straight line.

Let AB (Fig. 113) be a straight line, and AH, BK two perpendiculars from its extremities to the plane MN. Join H and K; and AH, BK, being parallel (129, *Cor.* i.), AH, BK, AB, HK, are all in the plane of the lines AB, AH (122). A perpendicular to MN from any point C in AB is parallel to AH, therefore it must meet the plane at some point D in the line HK (123). Consequently the straight line HK is the projection of AB on the plane MN.

Fig. 113.



PROBLEMS.

131. I. *To draw a perpendicular to a plane from a given point outside the plane.*

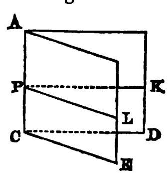
Find any three points B, E, H (Fig. 111), in the plane equidistant from the given outside point A; draw the circumference of a circle through these three points; and the line AP which joins the centre of the circle and the point A is the perpendicular required. For, since some line through A must be perpendicular to the plane, if a circle be described, with the foot of the perpendicular as centre, and the distance of one of the points B, E, H, as radius, this circle must pass through the other two (128, *Cor.* ii.); therefore the perpendicular from A to the plane must coincide with a line joining A and the centre of a circle which passes through B, E, and H (37, II., *Cor.* i., *Axiom* 1).

II. *To draw a perpendicular to a plane at a given point in the plane.*

Draw, by the last, from any assumed point outside the plane a perpendicular to the plane, and a line through the given point parallel to this perpendicular will be the perpendicular required (129).

III. *To draw a plane at right angles to a given line at a given point in the line.*

Fig. 114.



Let P be the given point in the given line AC (Fig. 114). Draw two planes AD and AE, so as to intersect each other in AC, by passing each of them through any two points in it. Also draw KP perpendicular to AP in one of the planes, and LP perpendicular to AP in the other plane. The common plane of the two lines PK and PL is the required plane (125).

132. *Def.* A line is said to be *oblique* to a plane when it makes unequal angles with any line in the plane passing through the meeting point.

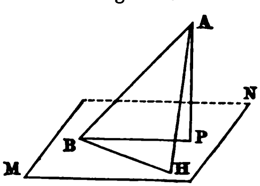
The *least angle* which an oblique line meeting a plane makes with any line in the plane is agreed on as the angle which the oblique line makes with the plane.

THEOREM.

133. *Of all the angles which an oblique line meeting a plane makes with lines in the plane, the least is that which it makes with its projection on the plane.*

Let AB (Fig. 115) be an oblique line meeting the plane MN. Draw AP perpendicular to the plane, and join B and P. BP is the projection of the line AB on the plane MN (130). Let BH be any other line in the plane MN passing through B; make BH = BP, and draw AH. The two triangles APB and AHB have AB common, BP = BH, but AH > AP (125, *Cor.* ii.); therefore the angle ABH > ABP (61).

Fig. 115.



Cor. i. Hence the angle which an oblique line makes with a plane is sometimes defined as the angle which it makes with a line in the plane joining the meeting point and the point where a perpendicular from the oblique line to the plane meets the plane.

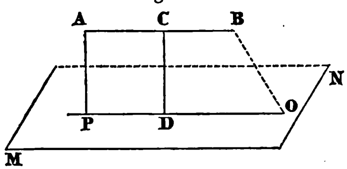
Cor. ii. A line passing through a plane makes equal vertical angles with the opposite sides of the plane.

134. *Def.* A line and a plane are parallel when being produced indefinitely they do not meet.

THEOREM.

A line is parallel to a plane when a perpendicular to the line is perpendicular to the plane.

Fig. 116.



Let AP (Fig. 116) be perpendicular to the straight line AB and to the plane MN; then if AB be not parallel to MN, let them when produced meet in O. Draw OP; and the angles OPA and OAP are both right angles, which is absurd.

THEOREM.

135. *A line is parallel to a plane when two perpendiculars from it to the plane are equal.*

Let AP and CD (Fig. 116) be two equal perpendiculars from the straight line AB to the plane MN, and AB is parallel to MN. Draw PD; and since AP and CD are parallel (129, *Cor.* i.), PD and AC are in the plane of AP and CD (122), and also parallel (67). Consequently AP is perpendicular to the plane MN and to the line AB (31), which is therefore parallel to the plane MN (134).

136. The truth of the following theorems is sufficiently obvious.

I. A line, as AB (Fig. 116), is parallel to a plane MN if it is parallel to any line PD in the plane MN; for to meet the plane MN evidently it must meet it in PD when sufficiently produced.

II. If AB be parallel to the plane MN, the intersection of a plane in which AB is situated with MN will be parallel to AB (30, *Def.*).

III. AB being parallel to the plane MN, a line through any point in MN parallel to AB must lie entirely in the plane MN (123, *Cor.* iv.).

SECTION II.

RELATIVE POSITION OF PLANES.

PARALLEL PLANES.

Def. Two planes are parallel which, though produced indefinitely, do not meet.

THEOREM.

137. *Two planes are parallel if a line between them is perpendicular to both.*

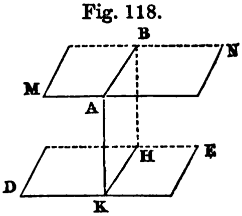
For if the planes meet when sufficiently produced, two lines, one in each of the planes, may be drawn from any point in their intersection at right angles to the common perpendicular (124), which is impossible (21).

THEOREM.

138. *The lines of intersection of the same plane with two parallel planes are parallel.*

Let MN and DE (Fig. 118) be two parallel planes cut by the plane AH; then AB and KH, the intersections of AH with MN and DE respectively, are parallel.

Fig. 118.



For AB and KH are in the same plane AH, and cannot meet though indefinitely produced, since the planes MN and DE cannot meet; therefore AB and KH are parallel (30).

THEOREM.

139. *Parallel lines between parallel planes are equal.*

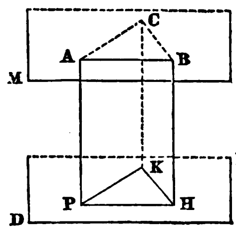
For if AK and BH (Fig. 118) be two parallel lines between the parallel planes MN and DE, then AB and KH,

the intersections of their common plane with MN and DE respectively, shall be parallel (138). Therefore ABHK is a parallelogram, and AK = BH (66).

Cor. i. Two parallel planes are at the same distance from each other at every point; for perpendiculars from any two points in either plane to the other are parallel (129, *Cor.* i.), and therefore equal.

Cor. ii. Two planes are parallel when they intercept between them three equal parallel lines which are not in the same plane. For if AP, BH, and CK (Fig. 119) be three equal parallel lines intercepted between the planes MN and DE, then a plane through A parallel to DE must pass through B and C (139), and consequently must coincide with MN (123).

Fig. 119.

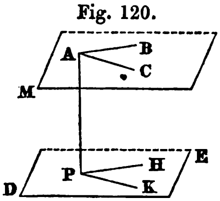


THEOREM.

140. *A straight line perpendicular to one of two parallel planes is perpendicular to the other.*

Let MN and DE (Fig. 120) be two parallel planes, and AP a straight line perpendicular to the plane DE. In the plane DE draw any two lines PH, PK, through the point P; and let the plane of the lines PH, PA, cut the plane MN in the line AB; also, let the plane of the lines PK, PA, cut MN in the line AC. Then AB and AC are respectively parallel to PH and PK (138). Therefore AP being perpendicular to PH and PK (124), is perpendicular to AB and AC (31), and consequently to the plane MN (125).

Fig. 120.



Cor. i. Through the same point only one plane can pass so as to be parallel to a given plane (127).

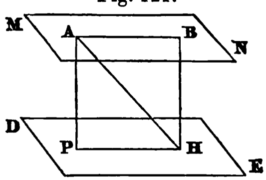
Cor. ii. Two planes parallel to a third are parallel to each other (137).

Theorem.

141. *The projections of an oblique line intercepted between two parallel planes on the two planes are equal and parallel.*

Let AH (Fig. 121) be an oblique line intercepted between the two parallel planes MN and DE. Draw HB perpendicular to MN, and AP perpendicular to DE. Then AB and HP are the projections of AH on the planes MN and DE respectively (130), and since AP and BH are parallel (129, *Cor.* i.), being both perpendicular to each of the planes (140), AB and PH are parallel (138) and equal (66).

Fig. 121.



Cor. Hence a straight line passing through two parallel planes makes with them equal alternate interior, equal alternate exterior, and equal corresponding angles (132, 133).

Theorem.

142. *When three or more parallel lines which are not in the same plane meet two parallel planes, lines drawn in each of the planes to join the corresponding points of meeting form equal triangles, or equal polygons, on the planes.*

Let AP, CK, and BH (Fig. 119) be three parallel lines intercepted between the parallel planes MN and DE. Draw AC, PK, AB, PH, BC, HK; and since AP and BH are parallel, AB and PH are parallel (138) and equal (66); so also are AC and PK, BC and HK. Therefore the triangle ABC is equal to the triangle PHK (57).

When there are more than three parallel lines the equality of the polygons may be shown by drawing corresponding diagonals which will divide the polygons into the same number of triangles, equal two and two.

THEOREM.

143. *Two angles in different planes which have their sides respectively parallel and opening in the same direction are equal; and their planes are parallel.*

Let BAC and HPK (Fig. 119) be two angles having AB and AC respectively parallel to PH and PK. Take AB = PH, AC = PK; and since AB is parallel to PH, BH is parallel and equal to AP (67, 66). For the same reason CK is parallel and equal to AP; therefore CK and BH are equal and parallel (129, *Cor.* ii.). Consequently the triangle ABC is equal to the triangle PHK (57), and the angle BAC to the angle HPK.

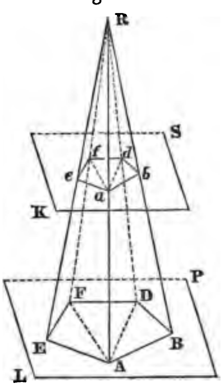
Also, since AP = BH = CK, the plane of the triangle PHK is parallel to the plane of the triangle ABC (139, *Cor.* ii.).

THEOREM.

144. *When three or more straight lines pass through two parallel planes and meet at the same point, lines joining the corresponding points of intersection form similar triangles or similar polygons on the two parallel planes.*

Let LP and KS (Fig. 123) be parallel planes, and RA, RB, RD, RF, RE, lines passing through them from the same point R; then the figure *abdfe* is similar to ABDFE.

Fig. 123.



For since *ab* is parallel to AB (138), *bd* to BD . . ., the triangle *a*R*b* is similar to the triangle ARB (81, *Cor.*), *b*R*d* to BRD. . . . Therefore (86) AB : *ab* : : BR : *b*R : : BD : *bd* : : DR : *d*R : : DF : *df* . . . . ; and, omitting the even ratios, AB : *ab* : : BD : *bd* : : DF : *df* : : FE : *fe* : : EA : *ea*.

Also, because BA, BD are respectively parallel to *ba*, *bd*, the angle ABD = *abd* (143); and, for the same reason, BDF = *bdf*, DFE = *dfe*. . . .

Consequently the two polygons are similar (94, *Def.*).

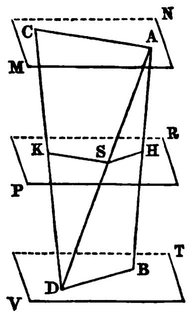
Cor. Any number of lines proceeding from a point are cut proportionally by two parallel planes (86).

THEOREM.

145. *The parts of two lines intercepted between three parallel planes are proportional.*

Let MN, PR, and VT (Fig. 124) be three parallel planes cutting the two lines AB and CD in the points A, H, B, and C, K, D, respectively. Join the points A and D by a line AD cutting the plane PR in the point S, and draw AC, BD, HS, SK. The triangles ADC and SDK, BAD and HAS, are similar, since SK is parallel to AC, and HS to BD (138).

Fig. 124.



Therefore

$$AH : HB :: AS : SD :: CK : KD;$$

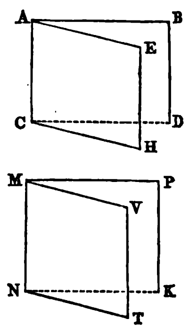
or, omitting the middle ratio,

$$AH : HB :: CK : KD.$$

PLANES WHICH INTERSECT.

146. The inclination of two planes which intersect is called an angle, or, for distinction sake, a *dihedral* angle. The two planes are the *faces*, and the line of intersection is the *edge* of the dihedral angle.

Fig. 125.



A dihedral angle, such as that formed by the two planes EC and BC (Fig. 125), is represented by four letters, HACD, the second and third letters being placed at the extremities of the edge.

Two dihedral angles are *equal* when the edge and faces of one can be made to coincide with the edge and faces of the other respectively.

Two lines, one in each of the faces of any dihedral angle, and each drawn perpendicular to the edge at the same point, form a plane angle, which we shall call *the angle corresponding to* the dihedral angle. Thus the angle HCD (Fig. 125) is the plane angle corresponding to the dihedral angle HACD, HC and DC being both perpendicular to AC.

The angle corresponding to any dihedral angle will be the same at whatever point along the edge the two perpendiculars be drawn, for two perpendiculars to AC at any other point than C will be respectively parallel to HC and DC (30, I.); consequently they will contain an angle equal to the angle HCD (143).

THEOREM.

147. *Two dihedral angles are equal when their corresponding plane angles are equal, and conversely.*

If HCD and TNK (Fig. 125) be the plane angles corresponding respectively to the two dihedral angles HACD and TMNK, and HCD = TNK; then HACD = TMNK.

For if AC be made to coincide with MN, the point C falling on N, and the plane AD revolve round AC in this new position, it will finally coincide with the plane MK, and CD at the same time will coincide with NK (14). Also CH will be in the plane of the lines NK, NT (127, *Cor.*); therefore it must coincide with NT, since HCD = TNK. Consequently the plane of the lines AC, CH, will coincide with the plane of the lines MN, NT; that is, the planes HA and TM will coincide, and the two dihedral angles HACD and TMNK will, therefore, be equal (146).

The converse also is obviously true.

THEOREM.

148. *Any two dihedral angles are related to each other as their corresponding plane angles.*

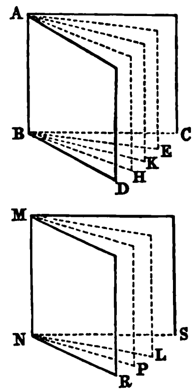
Let DBC and RNS (Fig. 126) be two plane angles corresponding to the two dihedral angles DABC and RMNS.

Then DABC : RMNS : : DBC : RNS.

Divide DBC into n equal angles by lines BE, BK, BH, ... , and RNS into m angles, each equal to EBC, by lines NL, NP, . . .

The planes of the lines AB and BE, AB and BK, AB and BH, ... MN and NL, MN and NP, . . . divide the dihedral angles DABC and RMNS respectively into n and m dihedral angles each equal to EABC (147).

Fig. 126.



Therefore

$$\frac{\text{DABC}}{\text{EABC}} = n = \frac{\text{DBC}}{\text{EBC}};$$

and

$$\frac{\text{RMNS}}{\text{EABC}} = m = \frac{\text{RNS}}{\text{EBC}}.$$

Consequently DABC : RMNS : : DBC : RNS.

We have supposed that the angles DBC and RNS are commensurable. When this is not the case it may be shown, by reasoning as in No. 7, that the proportion still holds.

Cor. i. Since $\frac{\text{DABC}}{\text{EABC}} = \frac{\text{DBC}}{\text{EBC}}$, if EBC be the angular unit, and the corresponding dihedral angle EABC be taken as the unit dihedral angle, then $\frac{\text{DABC}}{\text{unit dihedral angle}}$ = DBC ; that is, the number which expresses how often DBC contains the plane angular unit will also express how often DABC contains the dihedral angular unit. The plane angle DBC is, therefore, a *measure* of the dihedral angle DABC.

A dihedral angle is said to be right, acute, or obtuse, according as its corresponding plane angle is right, acute, or obtuse.

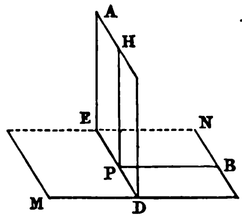
149. It is easy to infer from the preceding that when two planes intersect, the vertically opposite dihedral angles are equal; also that a plane cutting two parallel planes makes with them equal alternate interior, equal alternate exterior, and equal corresponding dihedral angles.

THEOREM.

150. *Two planes are perpendicular to each other when a line in either plane is perpendicular to the other.*

If the line HP in the plane AD (Fig. 127) is perpendicular to the plane MN, the plane AD is perpendicular to MN.

Fig. 127.



Let DE be the intersection of the two planes, and draw PB in the plane MN perpendicular to DE.

Then, since HP is perpendicular to the plane MN, HPB is a right angle, and measures the dihedral angle formed by the two planes, which is, therefore, a right angle (148, *Cor.* i.).

THEOREM.

151. *When two planes are perpendicular to each other, a line in either perpendicular to the line of intersection is also perpendicular to the other plane.*

Let the plane AD (Fig. 127) be perpendicular MN, and draw HP in AD perpendicular to DE the line of intersection with MN; also draw PB in MN perpendicular to DE. Then since the dihedral angle made by the planes is a right angle, the plane angle HPB which measures it is also a right angle (148, *Cor.* i.). Therefore HP being perpendicular to PE and PB, is perpendicular to the plane MN.

Cor. i. Since HP is the only perpendicular from H to the plane MN (125, *Cor.* i.), hence

When two planes intersect at right angles a perpendicular to one of the planes from any point in the other must lie entirely in the other plane.

Cor. ii. *When two planes are perpendicular to a third, their line of intersection is perpendicular to the third;* for a perpen-

dicular to the third plane from any point in the intersection of the other two must lie entirely in each of the others. Consequently it must coincide with their line of intersection.

SOLID ANGLES.

152. When three or more planes meet in a point they form a *solid* angle.

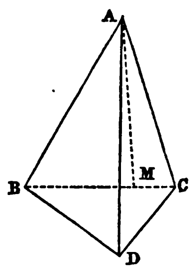
A solid angle is called *trihedral* or *polyhedral* according to the number of planes which intersect.

THEOREM.

153. *In every trihedral angle any two of the plane angles are together greater than the third.*

Let the plane angles BAD, DAC, and BAC (Fig. 128) form a trihedral angle at A; then BAD + DAC > BAC.

Fig. 128.



When BAC is equal to, or less than, either of the other two, the truth of the theorem is evident.

Let, then, BAC be the greatest plane angle, and take a part of it BAM = BAD, make AM = AD, and through M draw a line BC cutting the edges AB and AC in the points B and C; draw also BD and DC. The two triangles BAD and BAM are equal (54), therefore BD = BM. But BD + DC > BM + MC; consequently DC > MC. Hence in the triangles ACD and ACM, having AC common, AD = AM, and DC > MC, the angle CAD > CAM (61). Therefore BAD + DAC > BAC.

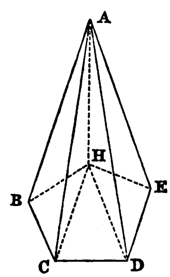
Cor. By dividing a polyhedral angle into trihedral angles it may be shown that any plane angle of a polyhedral angle is less than the sum of all the others.

Theorem.

154. *The sum of all the plane angles which form a solid angle is less than four right angles.*

If a plane cut the edges AB, AC, AD, ... of the polyhedral angle at A (Fig. 129) in the points B, C, D, ... , lines joining these points will form a polygon BCDEH on the plane.

Fig. 129.



Let S = sum of the interior angles of the polygon BCDEH;

n = number of sides in the polygon;

r = one right angle;

V = sum of plane angles at A;

B = sum of plane angles at the bases of all the triangles BAC, CAD, ...

Then $S + 4r = n \times 2r$ (72) $= V + B$ (46).

But from the trihedral angles formed at B, C, D . . . , HBC < ABH + ABC, BCD < ACB + ACD (153), and so on. Therefore B > S; consequently, since $S + 4r = V + B$, $V < 4r$; that is, the sum of the plane angles forming the polyhedral angle at A is less than four right angles.

Exercises.

1. When perpendiculars are drawn to a plane from different points in a line oblique to the plane, the feet of all the perpendiculars lie in the same straight line.

2. Through the same point only one plane can pass so as to be parallel to a given plane.

3. If parallel planes are intersected by parallel planes all the lines of intersection are parallel.

4. When a straight line is parallel to each of two planes which meet, it is parallel to their line of intersection.

5. Through a given point to draw a plane parallel to any two given lines.

6. A plane being drawn to bisect a line perpendicularly, any point in the plane is equidistant from the extremities of the line; and all points equidistant from the extremities of the line are situated in that plane.

7. A straight line meeting a plane is perpendicular to it when it makes equal angles with any three lines in the plane.

8. When a line is oblique to a given plane there is only one plane containing it that is perpendicular to the given plane.

9. Two triangles are similar in which the three sides are respectively parallel even when the triangles are in different planes.

10. Lines joining the extremities of three equal parallel lines which are not in the same plane form two equal triangles whose planes are parallel.

11. Through any two lines not in the same plane, two, and only two, planes can pass which are parallel.

12. To draw a line perpendicular to any two straight lines.

13. When a line is equally inclined to the two faces of a dihedral angle the points in which it meets them are equidistant from the edge.

14. Find the locus of a point equidistant from three given points.

15. The locus of a point equidistant from the faces of a dihedral angle is a plane bisecting the angle.

16. The angle corresponding to any dihedral angle is equal to, or the supplement of, the angle formed by perpendiculars drawn one to each of the planes at the same point of the line of intersection.

17. If three points be taken in the edges of a trihedral angle equidistant from the vertex, and a plane be passed through them, the foot of a perpendicular to this plane from the vertex will be the centre of a circle passing through the three points in the edges.

18. Planes bisecting the three dihedral angles of a trihedral angle all intersect in the same straight line.

19. Three planes respectively perpendicular to the three faces of a trihedral angle and bisecting the plane angles, intersect in the same straight line.

20. The extremities of either diagonal of a parallelogram are equidistant from any plane in which the other diagonal is situated.

PART IV.

ON SOLIDS.

DEFINITIONS.

A SPACE entirely enclosed by surfaces is called a *solid*.

Hence every solid has length, breadth, and depth or thickness (*Def.* I. Part I.).

The smallest number of plane surfaces necessary to form a solid is four; for the angular space intercepted by three planes which intersect in a point is not completely enclosed unless each of the planes is cut by a fourth.

155. I. Solids get different names according to the number and position of their bounding surfaces.

A tetrahedron is formed by four plane surfaces; a *pentahedron* by five; a *hexahedron* by six, and so on. *Polyhedron* is a general name for solids bounded by any number of plane surfaces. The planes are called the *faces*, and their intersections the *edges* of the polyhedron. *Diagonals* are straight lines joining the vertices of any two non-adjacent solid angles in a polyhedron.

II. A solid is said to be *regular* when all its solid angles are equal, and all its faces are equal regular polygons.

There can be only five regular polyhedrons; for, since the sum of all the plane angles which form a solid angle must be less than 360°, or four right angles (154), and three plane angles at least are necessary, the solid angles of a regular polyhedron can only be formed by the angles of

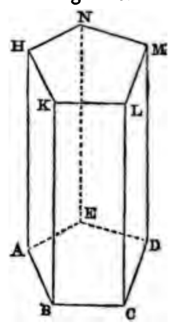
3, 4, or 5 equilateral triangles, 3 squares, or 3 regular pentagons (72, *Cor.* ii.).

It may be shown that the five regular polyhedrons are a tetrahedron, hexahedron, octahedron, dodecahedron, and icosahedron, containing respectively 4, 6, 8, 12, and 20 faces. In the tetrahedron, octahedron, and icosahedron, the faces are triangular; they are squares in the hexahedron, and pentagons in the dodecahedron.

III. A *prism* is a polyhedron bounded by two equal and parallel plane figures called *bases*, and as many parallelograms as there are sides in either base.

Thus if ABCDE and HKLMN (Fig. 130) are two equal and parallel polygons, and the lateral surfaces AK, BL, CM, . . . are parallelograms, the solid space contained by all these plane figures is a prism.

Fig. 130.

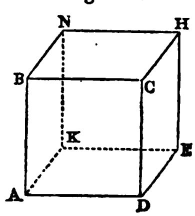


A prism may be formed by passing three or more parallel lines, which are not in the same plane, through two parallel planes, and drawing lines in each of the planes to join the corresponding points of intersection. The equal polygons that will be formed on the two planes (142) will be the bases, and the parallel lines between them will be the lateral edges of the prism.

IV. The *altitude* of a prism is a perpendicular between the two bases.

A lateral edge, or any line parallel to it between the bases, is called the *length* of a prism; and a prism is *right* or *oblique* according as its altitude and length are parallel or inclined to each other.

Fig. 131.



V. When the bases of a prism are parallelograms it is called a *parallelopiped* (Fig. 131).

The six faces of a parallelopiped are parallel and equal, two and two; for since AD = KE = NH = BC (66), and these lines are all parallel (129, *Cor.* ii.), the parallelogram BK is parallel to CE (139, *Cor.* ii.),

and is also equal to it (142). For a similar reason AC is parallel and equal to KH. Hence any face of a parallelopiped may be regarded as one of its bases.

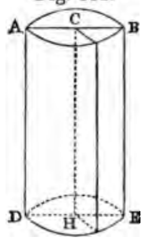
A *rectangular* parallelopiped is a right prism whose bases, and lateral faces consequently, are rectangles; when the six faces are squares it is called a *cube*.

VI. A prism is *triangular*, *quadrangular*, . . . according to the number of sides in its bases. When the bases are regular polygons having circles described about them, and the number of sides in each is indefinitely increased, the polygons become circles (96, *Def.*), and the *limit* of the prism is a *cylinder*. Hence

VII. A cylinder is a prism with circular bases (Fig. 132).

A line, as CH, joining the centres of the two bases is the *axis* of the cylinder. When the axis is perpendicular to the two bases the cylinder is a *right cylinder*. It may be conceived as generated by a rectangle CE revolving round one of its sides, CH, which remains fixed.

Fig. 132.



VIII. When lines are drawn from a point outside any plane to the vertices of the angles of a polygon described on the plane, the solid contained by the polygon and the triangles thus formed is called a *pyramid* (Fig. 129).

The polygon BCDEH is the *base*, and the point A from which the lines are drawn is the *vertex* of the pyramid.

IX. The *altitude* of a pyramid is a perpendicular from the vertex to the plane of the base.

In a pyramid whose base is a regular plane figure a line from the vertex to the centre of the base is called the *axis;* and a pyramid is *regular* when its axis and altitude coincide. A tetrahedron is the only regular pyramid which can be also a regular solid (II.).

A straight line from the vertex of a pyramid perpendicular to the base of any of the lateral triangles is called the *apothem*.

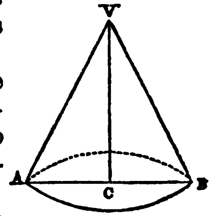
X. From the number of sides in its base a pyramid is called *triangular*, *quadrangular*. . . .

If the base of a pyramid be a regular polygon inscribed in a circle, and the number of its sides be indefinitely increased, the base will approach to an equality with the circle (96, *Def.*), and the *limit* of the pyramid will be a *cone.* Hence

XI. A cone is a pyramid with a circular base (Fig. 133).

A *right* cone has its axis perpendicular to its base, and may be conceived as generated by a right-angled triangle VCB revolving round one of the small sides VC which remains fixed.

Fig. 133.



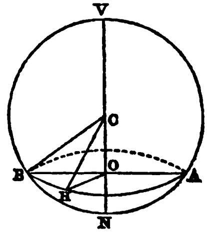
The *apothem* of a cone is a straight line from the vertex to any point in the circumference of the base. It is equal to the hypothenuse VB of the generating triangle.

XII. A *sphere* is a solid bounded by a curved surface, every point of which is equidistant from a point within it called the *centre.*

A sphere may be conceived as generated by a circle revolving round any of its diameters as an axis.

Every plane section through a sphere is a circle; for if a diameter VN (Fig. 134) be drawn perpendicular to the plane section AHB, the point O, at which it meets it, will be equidistant from every point in the curve AHB, since the lines OB, OH, . . . drawn from O to different points in the curve form with CO and the radii CB, CH, . . . the equal right-angled triangles COB, COH, . . .

Fig. 134.



Plane sections through the centre of a sphere are called *great circles of the sphere; small circles* are those which do not pass through the centre.

Two great circles of a sphere bisect each other in a common diameter.

A plane which touches a sphere in one point only is called a *tangent plane.* A radius to the point of contact is normal

to the plane, for it is the shortest line that can be drawn from the centre to the plane (125, *Cor.* ii.).

XIII. Solids generated by the motion of a plane figure round a fixed axis are called *solids of revolution.*

The sphere, right cylinder, and right cone are solids of revolution; they are also sometimes called the *three round bodies.*

SECTION I.

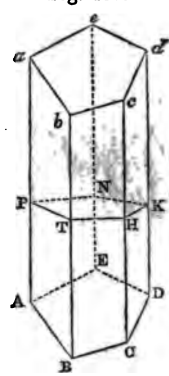
SURFACES OF SOLIDS.

THEOREM.

156. *The lateral surface of a prism is measured by the product of one of its lateral edges and the perimeter of a section perpendicular to it.*

Let ABCd (Fig. 135) be a prism, and PTHKN a section perpendicular to the edge Aa; it is also perpendicular to each of the other edges Bb, Cc, Dd, ..., since they are all parallel (155, III., 129, *Cor.* ii.). Hence, regarding the lateral edges of the prism, which are all equal (139), as the bases of the lateral parallelograms, the sides of the section PTHKN become their altitudes respectively, and the sum of the surfaces of the parallelograms, that is, the lateral surface of the prism, is measured by the product of one of the bases by the sum of their altitudes (106, *Cor.* ii.).

Fig. 135.



Therefore, lat. surf. of prism
$= Aa \times \overline{PT + TH + HK + KN + NP}.$

Cor. i. When the prism is a right prism Aa is equal to the altitude (139), and the section PTHKN is parallel and equal to the base ABCDE (137, 142); hence,

The lateral surface of a right prism is measured by the product of its altitude and the perimeter of its base.

Cor. ii. The product of the axis and the perimeter of a section perpendicular to it measures the lateral or convex surface of a cylinder; for a cylinder may be regarded as a prism having circular bases (155, VI.), and its axis equal to the length.

When the cylinder is a right cylinder its convex surface is measured by the product of its axis and the perimeter of its base.

Cor. iii. In a right equilateral cylinder, which has the axis equal to the diameter of the base, the convex surface is four times the surface of the base. For, representing the two surfaces by L and B respectively, and calling the radius of the base r,

$$L = 4\pi r^2, \text{ by the last,}$$

and

$$B = \pi r^2 \ (110).$$

Therefore

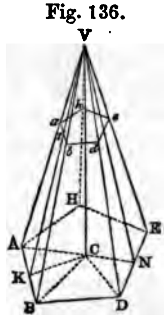
$$L = 4B.$$

Hence, adding 2B to both sides, the total surface of such a cylinder is six times the surface of the base.

THEOREM.

157. *The lateral surface of a regular pyramid is measured by half the product of the apothem and the perimeter of the base.*

Fig. 136.



As the axis VC (Fig. 136) of the regular pyramid ABDV is perpendicular to the base, the triangles VCA, VCB, VCD, . . . are equal (54). Therefore the edge VA = VB = VD = . . . and the lateral triangles are isosceles and equal (57). Also the apothem VK = VN = . . ., and these are the altitudes of the lateral triangles. Hence the sum of the surfaces of the triangles, that is, the lateral surface of the pyramid, is measured by half the product of the apothem and the sum of the bases of

the triangles which is the perimeter of the base of the pyramid.

A section parallel to the base and bisecting the altitude is a polygon similar to the base (144), and has a perimeter equal to half the perimeter of the base, as may easily be shown from the properties of similar triangles; therefore, the lateral surface of a regular pyramid is also measured by the product of the apothem and the perimeter of a section midway between the vertex and base. If the pyramid be not a regular one, the lateral triangles, not having an equal altitude, must be measured separately and their areas added together.

Cor. i. Since a right cone may be regarded as a regular pyramid having a circular base (155, XI.), its lateral or convex surface is measured by half the product of the circumference of its base and the apothem; or, by the product of the apothem and the perimeter of its middle section.

Cor. ii. When a regular pyramid and a right prism have the same or equal bases, and the apothem of the pyramid is equal to the altitude of the prism, the surface of the prism is double the surface of the pyramid.

The same is true of a right cylinder and a right cone.

Cor. iii. In an *equilateral* cone the apothem is equal to the diameter of the base, and the lateral surface is double the surface of the base. For calling the two surfaces L and B respectively, and the radius of the base r,

$$L = 2\pi r^2,$$

and

$$B = \pi r^2.$$

Therefore

$$L = 2B.$$

Also $L + B = 3B$; that is, the total surface is three times the surface of the base.

THEOREM.

158. *When a regular pyramid is truncated by a section parallel to the base, the lateral surface of the frustum is measured by half the sum of the perimeters of the upper and lower bases multiplied by the remainder of the apothem; or, by the product of the perimeter of its middle section and the remainder of the apothem.*

1°. Let ABDEHV (Fig. 136) be a regular pyramid cut by a section *abdeh* parallel to the base. The polygon *abdeh* is similar to the base (144), and has its sides respectively parallel to those of the base (138). Hence the lateral faces of the frustum *ab*BA, *bd*DB, . . . are trapeziums, and each trapezium has an altitude equal to *t*K, the remainder of the apothem. Therefore the sum of the areas of the trapeziums, that is, the lateral surface of the frustum, is measured by half the product of the remainder of the apothem and the sum of the perimeters of the parallel bases (108).

2°. From the properties of similar triangles (86) it may easily be shown that the perimeter of a section parallel to the bases and bisecting *t*K is equal to half the sum of the perimeters of the bases; hence,

The lateral surface of the frustum is also measured by the product of the perimeter of its middle section and the remainder of the apothem.

Cor. i. When a right cone is truncated by a section parallel to the base, the frustum has two circular bases (144), and its convex surface is measured by half the product of the apothem and the sum of the circumferences; or, by the product of the circumference of its middle section and the remainder of the apothem.

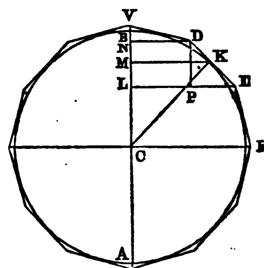
THEOREM.

159. *The surface of a sphere is measured by the product of one of its diameters and the circumference of a great circle.*

A regular polygon having an even number of sides being described about a circle, a diameter AB (Fig. 137)

may be drawn, which, when produced, will pass through V and X, the vertices of two of the angles, and bisect both the circle and the polygon. If this diameter remain fixed, the semicircle BKA revolving round it will generate a sphere, and the half polygon VDEFX will generate a solid, consisting of cones and truncated cones, formed by the revolution of the sides VD, DE, EF, . . . Draw a radius to K, the middle point of DE, which generates one of the truncated cones;

Fig. 137.



also, draw EL, KM, DN, perpendicular, and DP parallel to VX. The two triangles CMK and DPE are similar (83); therefore

$$CK : DE :: MK : DP,$$

and

$$\frac{CK}{MK} = \frac{DE}{DP} = \frac{DE}{NL}, \ldots (1).$$

The middle section of the truncated cone is described by MK as radius; calling the circumference of this circle c, and that of a great circle of the sphere C,

$$C : c :: CK : MK \text{ (96, } \textit{Cor.} \text{ ii.),}$$

and

$$\frac{C}{c} = \frac{CK}{MK} \ldots (2).$$

Comparing equations (1) and (2) we get

$$C \times NL = c \times DE \ldots (3).$$

But $c \times DE$ measures the convex surface of the truncated cone (158, *Cor.* i.). Therefore, from equation (3), the convex surface of each cone (157, *Cor.* i.), or truncated cone, is measured by the product of its altitude and the circumference of a great circle of the sphere; and the sum of the convex

surfaces of all the cones and truncated cones is measured by the product of the sum of their altitudes and the circumference of a great circle of the sphere. Consequently

$$\text{surface of entire solid} = \text{C} \times \text{VX} \quad . \; . \; . \quad (4).$$

If now the number of sides in the regular polygon VDEFX be conceived as indefinitely increased it will have the circle for its limit (96, *Def.*), VX will approach to an equality with BA, and the difference of the sphere and solid will become less than any given solid however small.

Hence, calling a diameter of the sphere D,

$$\text{surface of sphere} = \text{C} \times \text{D} = 4\pi r^2 \quad . \; . \; . \quad (5).$$

Cor. i. Calling S and *s* the surfaces of the sphere and of a great circle of the sphere respectively,

$$\text{S} = 4\pi r^2,$$

and

$$s = \pi r^2 \; (110);$$

therefore

$$\text{S} = 4s;$$

that is, the convex surface of a sphere is equal in area to four great circles of the sphere. Hence (110),

The surface of a sphere whose diameter is one foot is

$$\cdot 7854 \times 4 = 3\cdot 14159 \text{ square feet.}$$

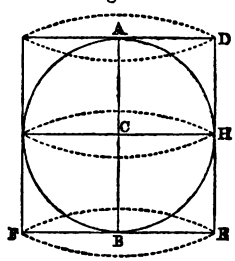
Cor. ii. The convex surface of a zone or spherical segment is measured by its altitude multiplied by the circumference of a great circle of the sphere; for the surface of a zone or segment is the limit of the convex surfaces of the circumscribed truncated cones having the same altitude. Thus the surface of the segment generated by the arc BK is measured by the product of the circumference of a great circle and BM the altitude of the segment; and C × ML measures the convex surface of the zone generated by the arc between MK and LE.

THEOREM.

160. *The surface of a sphere is equivalent to the lateral surface of an equilateral circumscribed cylinder.*

Let DF (Fig. 138) be a square circumscribed about the circle AHB which has its diameter AB parallel to DE, a side of the square. If AB remain fixed, and the semicircle AHB and half square AE revolve round it, the semicircle will generate a sphere, and the half square will generate an equilateral cylinder circumscribed about the sphere. The base of the cylinder is equal to a great circle of the sphere, and its lateral surface is measured by the circumference of its base multiplied by its altitude or axis, which is equal to a diameter of the sphere (156, *Cor.* ii.). Therefore the lateral surface of the cylinder and the convex surface of the sphere having the same measurement are equivalent.

Fig. 138.



Cor. i. When the two bases are added to the lateral surface of the cylinder, the total surface is equivalent to six great circles of the sphere. Hence the surface of a sphere is two-thirds of the total surface of the circumscribed cylinder.

THEOREM.

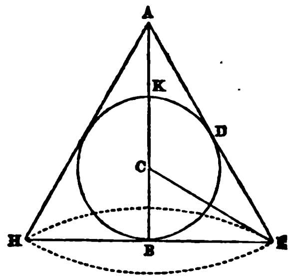
161. *The surface of a sphere is related to the total surface of a circumscribed equilateral cone as* 4 *to* 9.

Let BDK (Fig. 139) be a circle inscribed in the equilateral triangle AEH. If the perpendicular AB remain fixed, the right-angled triangle ABE revolving round it will generate an equilateral cone, and the semicircle BDK will generate a sphere inscribed in the cone.

Let S be the surface of the sphere, and s the area of a great circle of the sphere; S^1 the total surface of the cone, and s^1 the surface of its base; then,

$s : s^1 :: BC^2 : BE^2 = CE^2 - BC^2 :: 1 : 3.$
(117, *Cor.* ii. 74).

Fig. 139.



Therefore

$$s^1 = 3s = 3\pi r^2, \text{ (110)}.$$

But

$$S^1 = 3s^1 = 9\pi r^2 \text{ (157, } \textit{Cor.} \text{ iii.)},$$

and

$$S = 4\pi r^2 \text{ (159)}.$$

Consequently

$$S : S^1 :: 4\pi r^2 : 9\pi r^2 :: 4 : 9.$$

SECTION II.

VOLUMES OF SOLIDS.

162. The *solidity* or *volume* of a solid is the number of units of volume which it contains.

A cube constructed on the unit square, or, which is the same thing, having for its edge the unit of length, is agreed on as the unit of volume.

Two solids are *equal* when one being properly applied to the other, they fill exactly the same space; they are *equivalent* when they contain equal volumes. A prism and a pyramid may contain the unit of volume the same number of times, but cannot be made to fill the same space—they may be equivalent, but cannot be equal.

PRISM AND CYLINDER.

THEOREM.

163. *Two right prisms are equal when they have equal bases and equal altitudes.*

Let P and p be the two prisms; and let B, B′ be the bases of P; b, b' those of p, and A, a their respective altitudes. If the base b be properly applied to B, it will coincide with it, and the lateral edges of p will coincide respectively with those of P, since the prisms are right (126) and the altitudes equal. Therefore, the base b' must also coincide with B′, and the two prisms must coincide in every respect.

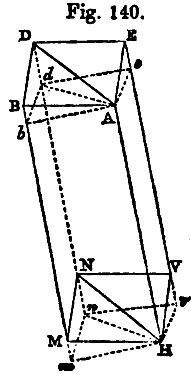
Cor. The theorem may also be extended, as is obvious, to two right cylinders (155, VII.).

THEOREM.

164. *If two planes be drawn perpendicular to one of the lateral edges of an oblique prism, and passing through its extremities, the lateral surfaces, when produced, will form with these planes a right prism equivalent to the oblique one.*

Let two planes be drawn through the points A and H (Fig. 140) perpendicular to the edge AH of the oblique prism MHVD; let the other edges of the prism meet the plane passing through A in the points b, d, e; and when produced let them meet the plane passing through H in the points m, n, v. The solid mHvd is a prism, since the polygons Hmnv and Abde are parallel (137), and equal (142), and the lateral surfaces Hb, He, ... are parallelograms, AH and bm being parallel, also Hm and Ab (138); and so of the others. Moreover MB = AH = mb = EV = ev . . .; therefore, taking away the common parts Mb, Ve, ... we get Mm = Bb, Ee = Vv, ...

Fig. 140.



If now the wedge-shaped solid A*bde*D be applied to H*mnv*N, the base A*bde* will coincide with H*mnv*, and since the lateral edges are perpendicular to these bases, the lines *b*B, *d*D, ... must fall on *m*M, *n*N, ... (126), and being equal to them will coincide with them respectively. Consequently, A*bde*D being taken from and added to the oblique prism MHVD, the volume of the latter remains unaltered; that is, MHVD is equivalent to the right prism *m*H*vd*.

Cor. i. If the base HMNV be a parallelogram, the base H*mnv* will also be a parallelogram, for the planes BH and DV being then parallel (155, V.), the lines H*m* and *nv* must be parallel (138), also *mn* and H*v*. Hence

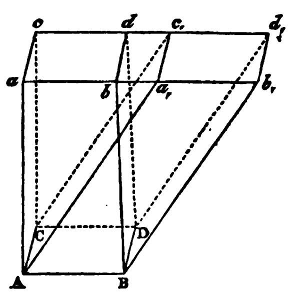
Cor. ii. *A diagonal plane divides every parallelopiped into two equivalent triangular prisms;* for if a plane be drawn through the extremities of the two parallel lines AH, D*n*, it will divide the right parallelopiped into two equal triangular prisms (163), and the oblique parallelopiped into two oblique triangular prisms respectively equivalent to the right prisms, and, therefore, equivalent to each other.

THEOREM.

165. *Two parallelopipeds are equivalent which have the same or equal bases and equal altitudes.*

Let the two parallelopipeds stand on the base ACDB (Fig. 141); then, since they have an equal altitude, the edges *ab*, $a_{\prime}b_{\prime}$, *ac*, $a_{\prime}c_{\prime}$, ... of the upper bases are all in a plane parallel to ACDB, and *ab*, $a_{\prime}b_{\prime}$ lie in the same straight line, or are parallel to each other.

Fig. 141.



1°. Let $ab_{\prime}$ be a straight line.

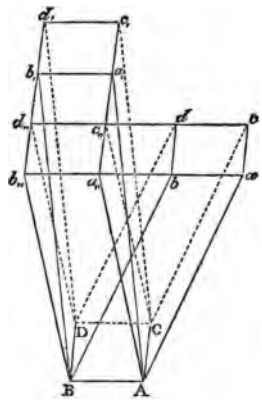
The triangle $Aaa_{\prime}$ is equal to $Bbb_{\prime}$ (57); also, the plane $Aab_{\prime}B$ makes equal corresponding dihedral angles with the parallel planes A*c*, B*d* (149); consequently, if the

triangular prism $Bbb_{,}d_{,}$ be applied to the prism $Aaa_{,}c_{,}$ so that the face Bd may coincide with its equal Ac, the base $Bbb_{,}$ will coincide with the base $Aaa_{,}$. Therefore, $bb_{,}$ coinciding with $aa_{,}$, the parallelogram $bd_{,}$ will coincide with its equal $ac_{,}$; and in like manner the remaining faces and edges of one will coincide with the remaining faces and edges of the other respectively: that is, the two triangular prisms are equal.

If now, from the entire solid which has AD and $ad_{,}$ for its bases, we take away, first, the prism $Aa_{,}ac$, and then the equal prism $Bb_{,}bd$, the two remainders, that is, the two parallelopipeds $ABDd$ and $ABDd_{,}$, must be equivalent.

2°. Let $ABDd$ and $ABDd_{,}$ (Fig. 142) be the two prisms—where none of the edges in the upper base of one is in the same straight line with an edge of the other. Then ab, cd being produced, are cut by $c_{,}a_{,}$, $d_{,}b_{,}$ when produced, in the points $c_{,,}$, $a_{,,}$, $d_{,,}$, $b_{,,}$, making the parallelogram $a_{,,}b_{,,}d_{,,}c_{,,} = abdc = a_{,}b_{,}d_{,}c_{,} = ABDC$. Hence the solid having ABDC and $a_{,,}b_{,,}d_{,,}c_{,,}$ for its lower and upper bases respectively, is a parallelopiped which, by the first case, is equivalent to $ABDd$, and also to $ABDd_{,}$. Consequently, $ABDd$ and $ABDd_{,}$ are equivalent to each other.

Fig. 142.



THEOREM.

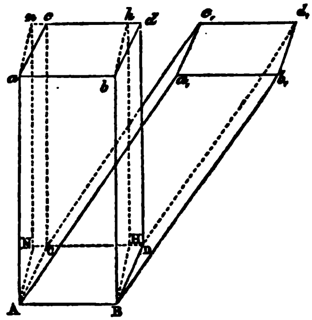
166. *Any parallelopiped may be changed into an equivalent rectangular one having an equal altitude and an equivalent base.*

When perpendiculars to the base of any *oblique* parallelopiped $ABDd_{,}$ (Fig. 143) are drawn through the vertices of the angles, and produced to meet the plane of the other

base produced, a *right* parallelopiped ABDd is formed equivalent to the oblique one (165).

Let, now, BH, AN be drawn perpendicular to AB, and bh, an perpendicular to ab; draw also the lines Hh, Nn.

Fig. 143.



The face ABba being regarded as the common base of the rectangular parallelopiped ABHh, and the right parallelopiped ABDd (155, V.), they have the same altitude, and therefore are equivalent (165). Consequently ABHh is equivalent to the given oblique parallelopiped.

THEOREM.

167. *Two rectangular parallelopipeds are related to each other as the products of their bases and altitudes.*

Let two rectangular parallelopipeds be called p and $p_{,}$, let b, $b_{,}$, be their respective bases, and a, $a_{,}$, their altitudes.

1°. Let $b = b_{,}$, and let a line l be contained m times in a, and n times in $a_{,}$; so that

$$l \times m = a,$$

$$l \times n = a_{,},$$

By drawing sections parallel to its base, and l distance asunder, p may be divided into m equal parallelopipeds, each having a height l and a base b (142, 163); and $p_{,}$ may be divided into n such parallelopipeds.

Hence, calling one of these small parallelopipeds v,

$$p = m \times v,$$

$$p_{,} = n \times v,$$

and, therefore,

$$p : p_{,} :: m : n :: m \times l : n \times l :: a : a_{,} ;$$

that is, *two rectangular parallelopipeds having equal bases are related to each other as their altitudes.*

We have supposed a and $a_{\prime}$ to be commensurable; when they are incommensurable, it may be shown by reasoning as in No. 104, that the proportion still holds.

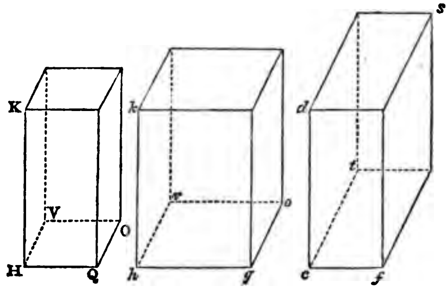


Fig. 144.

2°. Let there be two rectangular parallelopipeds KO and ko (Fig. 144) having equal altitudes; that is, HK $= hk$.

Assume a third rectangular parallelopiped cs, having its altitude $cd =$ HK, a side of its base $cf =$ HQ, the contiguous side $ct = hg$.

Considering KQ and df as the bases of KO and cs, HV and ct become their altitudes, and, by the first case,

$$\text{KO} : cs :: \text{HV} : ct \quad . \; . \; . \; (1);$$

also, dt and kg may be regarded as the bases of cs and ko, therefore

$$cs : ko :: cf : hv \quad . \; . \; . \; (2).$$

Comparing proportions (1) and (2) we get

$$\text{KO} : ko :: \text{HV} \times cf : hv \times ct :: \text{HV} \times \text{HQ} : hv \times hg :: \text{HO} : ho \; (106),$$

that is,

Two rectangular parallelopipeds having equal altitudes are related to each other as their bases.

If now P and P_1 be any two rectangular parallelopipeds whatsoever having B, B_1, A, A_1, for their bases and altitudes respectively, and if there be a third P_{11} having its base $\text{B}_{11} = \text{B}$, and its altitude $\text{A}_{11} = \text{A}_1$, then, by the first case,

$$\text{P} : \text{P}_{11} :: \text{A} : \text{A}_{11} = \text{A}_1,$$

and, by the second case,

$$P_{11} : P_1 : : B_{11} = B : B_1.$$

Therefore

$$P : P_1 : : A \times B : A_1 \times B_1.$$

Cor. i. Let b and c be the adjacent sides of B; b_1 and c_1 those of B_1; then (106)

$$P : P_1 : : A \times b \times c : A_1 \times b_1 \times c_1;$$

and, if

$$A_1 = b_1 = c_1 = \text{unit of length},$$

B_1 will be the unit square (106),
P_1 will be the unit cube (162),

and the proportion $P : P_1 : : A \times B : A_1 \times B_1$ will become $P : \text{unit cube} : : A \times B : 1$; or,

$$\frac{P}{\text{unit cube}} = A \times B:$$

that is, the number which expresses how often A contains the unit of *length* when multiplied by the number which expresses how often B contains the unit of *surface* will represent how often P contains the unit of *volume;* and this is briefly expressed by saying that *the volume of a rectangular parallelopiped is measured by the product of its base and altitude.*

Cor. ii. When P is a cube the equation above becomes $\frac{P}{\text{unit cube}} = A^3$. Hence *a cube is measured by the third power of one of its edges;* and in this way the term *cube* has come to designate the third power of any quantity whatsoever in arithmetic and algebra.

THEOREM.

168. *The volume of any prism, right or oblique, is measured by the product of its base and altitude.*

1°. Any parallelopiped may be reduced to an equivalent rectangular one having the same altitude and an equivalent base (166); therefore (167, *Cor.* i.),

The volume of any parallelopiped is measured by the product of its base and altitude.

2°. Any *triangular* prism is *one-half* of a parallelopiped having the same altitude and a *double* base (164, *Cor.* ii.); therefore,

The volume of any triangular prism is measured by the product of its base and altitude, since half the product of the base and altitude of the parallelopiped is equal to the product of the base and altitude of the prism.

3°. Any prism, such as ABCDN (Fig. 130), may be divided into triangular prisms, all having the same altitude, by passing planes through one of the edges, as BK, and each of the opposite edges NE, MD, . . .

Therefore, the sum of the bases of the triangular prisms multiplied by their common altitude, that is, the base of the entire prism ABCDN, multiplied by its altitude, will give its volume.

Cor. i. Since a cylinder may be regarded as a prism having circular bases (155, VII.); hence

The volume of a cylinder is measured by the product of its base and altitude.

If a be the altitude of a cylinder, and r the radius of its base,

$$\textit{volume of cylinder} = \pi r^2 a.$$

PYRAMID AND CONE.

THEOREM.

169. *If sections be made through a pyramid parallel to the base, their areas will be related to each other as the squares of their distances from the vertex.*

Let BDE and MNO (Fig. 145) be sections parallel to the base of a pyramid whose vertex is V and altitude VL. Also, let

$$\text{VH} = \text{A}, \ \text{VK} = a;$$

these are the respective distances of BDE and MNO from V (140, 125, *Cor.* ii).

Then BDE and MNO are similar figures (144), and (117)

$$BDE : MNO :: BD^2 : MN^2 :: BV^2 : MV^2 \text{ (86)}.$$

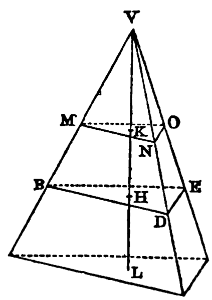
But, since the edges and altitude of a pyramid are cut proportionally (144, *Cor.*),

$$BV^2 : MV^2 :: A^2 : a^2.$$

Therefore

$$BDE : MNO :: A^2 : a^2.$$

Fig. 145.



Cor. If there be two pyramids having the same or equivalent bases and equal altitudes, sections parallel to the bases will be equivalent when made at equal distances from the vertices.

Let B, B_1, be the areas of the bases, b, b_1, the areas of sections parallel to the bases made at distances a, a_1, from the respective vertices, and A the common altitude of the two pyramids.

Then, by the theorem,

$$B : b :: A^2 : a^2,$$

and

$$B_1 : b_1 :: A^2 : a_1^2,$$

Hence, when $a = a_1$

$$B : B_1 :: b : b_1,$$

and since B and B_1 are equal in area, b and b_1 must also be equal in area.

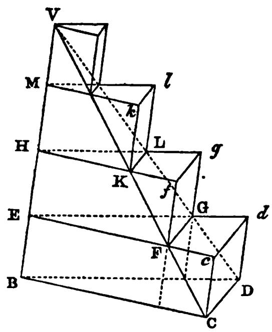
THEOREM.

170. *Two triangular pyramids having the same or equivalent bases and equal altitudes are equal in volume.*

1°. Let BCD (Fig. 146) be the base of one of the pyramids, V its vertex, and A its altitude. If the edge VB be

divided into n equal parts, and planes be drawn through the points of division parallel to the base, these planes will also divide the altitude into n equal parts (144, *Cor.*); let $\frac{A}{n} = a$. If now through the points C and D, F and G, and all the other points of intersection of the parallel planes with the edges VC and VD, lines be drawn parallel and equal to BE = EH = HM . . . , n triangular prisms, each having an altitude a, will be formed on the bases BCD, EFG, HKL, ... respectively, and the sum of these prisms will exceed the volume of the pyramid by the sum of the wedge-shaped solids FGdcCD, KLgfFG, ... This latter sum is less than the prism BCDd; for it is obviously less than the sum of prisms constructed on the bases FGdc, KLgf, . . . with cC, fF, . . . respectively as lateral edges; but the sum of these prisms is equivalent to the prism BCDd (168), as all have the same altitude a, and the sum of their bases FGdc, KLgf, ... is equal to BCD, since the edges fF, gG, kK, lL, ... if produced will divide BCD into figures respectively equal to FGdc, KLgf, ... (142). Hence the difference of the pyramid and the sum of the prisms BCDd, EFGg, HKLl, ... is less than the prism BCDd; and by increasing n indefinitely BCDd may be made less than any given solid. Therefore,

Fig. 146.



The pyramid BCDV *is the limit of the sum of the prisms* BCDd, EFGg, HKLl, ...

2°. Representing the pyramid BCDV by P, if, having assumed a second triangular pyramid P' with an altitude A and a base equivalent to BCD, one of its edges be divided into n equal parts, and prisms be formed as in the first case, the n prisms in the case of P' will be respectively equivalent to those formed in the case of P (168), for they will all have the same altitude a, and the bases two and two will be equivalent (169, *Cor.*). Hence, n being indefinitely increased,

P and P', which are the respective limits of the two equivalent series of prisms, must be equivalent or equal in volume.

A simpler demonstration of the above theorem, founded on the *method of indivisibles*, is sometimes given as follows:—

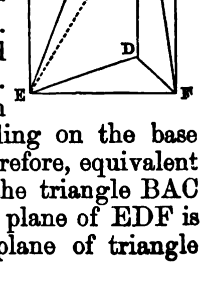
Every solid may be considered as made up of an indefinite number of elementary surfaces, each surface having an indefinitely small thickness. The pyramids P and P', since their altitudes are equal and their bases equivalent, will consist of the same number of such surfaces, equivalent two and two (169, *Cor.*). Hence, the sum of the surfaces composing P being equivalent to the sum of the surfaces composing P', the volume of P must be equal to the volume of P'. But it may be objected that however small the thickness of the component surfaces, they are in reality truncated pyramids, and the same difficulty must exist regarding the equivalence of any two of them that exists regarding the equivalence of the entire pyramids P and P'; or, if strictly mathematical surfaces, no number of them can form a pyramid (Part I., *Def.* II.).

THEOREM.

171. *A triangular prism may be divided into three pyramids having equal volumes.*

Let EDFBAC (Fig. 147) be a triangular prism. Draw the diagonals AF, AE, in two of the lateral parallelograms, and with the plane of these lines cut off the triangular pyramid EDFA.

Fig. 147.



The remaining solid may be regarded as a quadrangular pyramid having as base the parallelogram EBCF, and A for its vertex. In the base draw the diagonal EC, and with the plane of the lines EA, EC, divide the quadrangular pyramid into two triangular pyramids having EBC, EFC, as their respective bases, and a common vertex at A. These two pyramids having equal bases and equal altitudes have equal volumes (170). But the pyramid on the base EBC with vertex at A may also be regarded as standing on the base BAC and having its vertex at E, and is, therefore, equivalent to the first pyramid cut off (170); since the triangle BAC = EDF, and a perpendicular from A to the plane of EDF is equal to a perpendicular from E to the plane of triangle

BAC (139, *Cor.* i.). Therefore the triangular prism has been divided into three triangular pyramids EDFA, EBCA, EFCA having equal volumes.

Cor. i. A triangular pyramid may always be completed into a triangular prism having the same base and altitude but a triple solidity; for, if EDFA (Fig. 147) be a triangular pyramid, draw FC, and EB, parallel and equal to the edge DA, and join the points A, B, and C. The solid formed is a prism, since the triangle BAC is parallel and equal to EDF (139, *Cor.* ii., 142), and the lateral surfaces are, obviously, parallelograms; also, it has the same base and altitude as the pyramid, but, by the theorem, a triple volume.

Cor. ii. Since the volume of a prism is measured by the product of its base and altitude (168), hence

The volume of a triangular pyramid is measured by one-third of the product of its base and altitude.

Theorem.

172. *The volume of any pyramid is measured by one-third of the product of its base and altitude.*

If diagonals be drawn from any angle H (Fig. 129) in the base of a pyramid to the opposite angles C, D, . . . , the planes of the edge AH and each of these diagonals will divide BCDEHA into a number of triangular pyramids, all having the same altitude as the original pyramid. The volume of each triangular pyramid being measured by one-third of the product of its altitude and base (171, *Cor.* ii.), the sum of their volumes, that is, the volume of the pyramid BCDEHA, will be measured by one-third of the product of their common altitude and the sum of their bases, which is the base of the entire pyramid.

Cor. i. Every pyramid is one-third of a prism having the same base and altitude (168).

Cor. ii. Any polyhedron may be divided into pyramids and its volume determined by drawing lines from a point within it to the vertices of the solid angles formed by its bounding surfaces.

Cor. iii. Since a cone may be regarded as a pyramid having a circular base (155, XI.), hence

The volume of a cone is measured by one-third of the product of its base and altitude.

Calling the altitude of a cone a, and radius of base r,

$$\text{vol. of cone} = \tfrac{1}{3}\pi r^2 a.$$

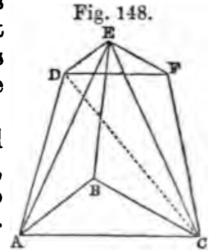
Cor. iv. When pyramids or cones have equal altitudes they are related to each other as their bases; when the bases are equal in area, they are related as their altitudes.

Theorem.

173. *A pyramid being truncated by a section parallel to the base, the volume of the frustum is equal to the sum of the volumes of three pyramids whose common altitude is the altitude of the frustum and whose bases are, respectively, those of the frustum and a geometric mean between them.*

Let ABCDEF (Fig. 148) be a triangular truncated pyramid having its bases ABC and DEF parallel. In the lateral trapeziums BD and BF draw the diagonals AE, CE; the plane of these lines will cut off a triangular pyramid whose base is ABC and vertex E. Let this pyramid be called p.

Fig. 148.



In the trapezium AF draw the diagonal CD, and the plane of the lines CD, CE, will divide the remaining solid into two other triangular pyramids having a common vertex at E, and the triangles CFD, CAD, respectively as bases. Let these pyramids be called p^1 and p^{11}, and (172, *Cor.* iv.)

$$p^1 : p^{11} : : \text{CFD} : \text{CAD} : : \text{DF} : \text{AC} \ (118) \quad . \ . \ . \ . \ (1).$$

But p^1 may also be regarded as standing on the base DEF with its vertex at C, and in this case (172, *Cor.* iv.),

$$p : p^1 : : \text{ABC} : \text{DEF} : : \text{AC}^2 : \text{DF}^2 \ (144, 115) \quad . \ . \ . \ (2).$$

Neglecting the middle ratios in proportions (1) and (2), and compounding

$$p : p^{11} : : AC : DF \ . \ . \ . \ (3);$$

and from proportion (1),

$$p^{11} : p^{1} : : AC : DF,$$

consequently

$$p : p^{11} : : p^{11} : p^{1};$$

that is, p^{11} is a geometric mean between p and p^{1}.

If, therefore, a pyramid be assumed equal in volume to p^{11} and having an altitude equal to the altitude of the frustum, its base must be a geometric mean between ABC and DEF, the respective bases of p and p^{1}, since each of the latter has an altitude equal to the altitude of the frustum (172, *Cor.* iv.).

Calling the volume of the frustum V, its altitude a, and its upper and lower bases b and B respectively,

$$V = \tfrac{1}{3} a \left(B + b + \sqrt{Bb}\right), \quad (172).$$

Any pyramid being given, another of equal volume having the same altitude and an equivalent triangular base may be constructed (172, *Cor.* iv.); and if both be truncated by sections parallel to, and equidistant from the bases, the pyramids cut off will also be equivalent; for they will have equal altitudes and equivalent bases (169, *Cor.*). Consequently the remaining solids or frusta will be equivalent.

Hence the theorem holds for all pyramids.

Cor. i. In the case of a truncated cone the radii of whose parallel bases are R and r, the formula given above becomes

$$V = \tfrac{1}{3} \pi a \left(R^2 + r^2 + Rr\right).$$

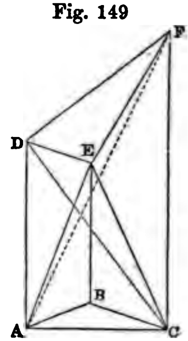
THEOREM.

174. *A triangular prism being cut by a plane oblique to the base, the volume of the frustum is equal to the sum of the volumes of three triangular pyramids having for their common base the base of the frustum, and for their respective vertices the points of intersection of the cutting plane with the edges of the prism.*

Let DEF (Fig. 149) be a section through a prism oblique to the base ABC.

1°. In the trapeziums BD, BF, draw the diagonals AE, CE; the plane of these lines will cut off a triangular pyramid having ABC for base and E for vertex.

Fig. 149



2°. Draw the diagonals CD, AF. The plane of the lines CD, CE, divides the remaining solid into two triangular pyramids having a common vertex at E, and CDF, CAD, as bases. Of these, the pyramid on the base CDF is equivalent to a pyramid on the base CAF with the vertex at B (170), for AD and CF are parallel, as are also BE and the plane CADF. But the pyramid on the base CAF with vertex at B may be regarded as standing on the base ABC with the vertex at F.

3°. The pyramid on the base CAD with vertex at E is equivalent to a pyramid on the same base with vertex at B, since, BE being parallel to the plane CAD, every point in BE is equidistant from the plane. But the pyramid on base CAD with vertex at B may be regarded as standing on the base ABC with its vertex at D.

Hence the frustum being equivalent to the sum of three pyramids having ABC for their common base and the points E, F, and D, for their respective vertices, is measured by one-third of the product of ABC and the sum of the perpendiculars let fall on it from the points E, F, and D.

Cor. When the prism is right, the volume of the frustum is measured by one-third of the product of the base and the sum of the edges.

THE SPHERE.

Theorem.

175. *The volume of a sphere is measured by the product of its convex surface and one-third of the radius.*

If three diameters, each perpendicular to the plane of the other two, be drawn in a sphere and tangent planes pass by their extremities, these planes will form a cube circumscribed about the sphere (137, 142, 155, V.). Lines drawn from the centre to the vertices of the solid angles will divide the cube into a number of pyramids, all having the centre for a common vertex, and the radius of the sphere for their altitude. Also the sum of the volumes of these pyramids will be measured by the product of the sum of their bases and one-third of the radius (172).

If now the solid angles of the cube be cut off by other tangent planes, and lines be drawn from the centre of the sphere to the vertices of the new solid angles thus formed, and if this process be conceived as repeated indefinitely, the circumscribed solid will approach indefinitely near to an equality with the sphere, the sum of the bases of its component pyramids differing from the convex surface of the sphere by an area less than any given area however small.

The sphere, therefore, being the limit of the circumscribed solid, has its volume measured by the product of its convex surface and one-third of the radius.

Calling the radius of the sphere r,

$$\text{its convex surface} = 4\pi r^2 \text{ (159)},$$

and

$$\text{volume of sphere} = 4\pi r^2 \times \tfrac{1}{3} r = \tfrac{4}{3}\pi r^3;$$

or, representing the diameter of the sphere by D,

$$\text{volume of sphere} = \tfrac{1}{6}\pi D^3.$$

A sphere, the diameter of which is one linear foot, contains $\frac{\pi}{6} = \cdot 5236$ cubic feet nearly.

Cor. i. From the demonstration it follows that the volume of a spherical sector is measured by the product of its convex surface and one-third of the radius; also that

The volumes of a sphere and of solids circumscribed about it are related to each other as their convex surfaces. Hence

Cor. ii. The volume of a sphere is related to the volume of a circumscribed equilateral cone as 4 : 9 (155, XI., 161).

Cor. iii. The volume of a sphere is to the volume of a circumscribed cylinder as 2 : 3 (155, VII., 160, *Cor.* i.); but this may be proved independently, as in the following

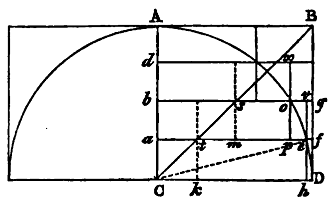
THEOREM.

176. *The volume of a sphere is two-thirds of the volume of the circumscribed cylinder.*

If a circle and a circumscribed square revolve round a fixed diameter which is parallel to one of the sides of the square, they will generate, respectively, a sphere and a circumscribed equilateral cylinder.

Let ACD (Fig. 150) be a quadrant, ABDC the corresponding part of the circumscribed square, AC one-half of the fixed diameter, and draw CB.

Fig. 150.



The quadrant revolving round AC will describe a hemisphere; the figure ABDC will describe a cylinder circumscribed about the hemisphere; and the right-angled triangle CAB will describe a right cone having a base and altitude equal respectively to those of the hemisphere and its circumscribed cylinder.

Let AC be divided into n equal parts, and through the

points of division draw the lines af, bg, ... perpendicular to AC; also, through the points of intersection of these lines with the arc AD, and the straight line CB, draw the lines hv, px, kt, ... parallel to AC.

The rectangles Cf, Ce, Ct, revolving round the fixed radius AC, describe three cylinders having a common altitude Ca; and the volume of the cylinder described by Cf is equivalent to the sum of the volumes of the cylinders described by Ce and Ct (168, *Cor.* i.); for the circle described by af is equivalent to the sum of the circles described by ae and at (119), since af = CD = Ce, and Ca = at, the triangle Cat being similar to CAB (81 *Cor.*), and, therefore, isosceles.

For a like reason the volume of the cylinder described by the rectangle ag is equivalent to the sum of the volumes of the cylinders described by ao and as; and so of the others.

Now the hemisphere is the limit of the sum of the cylinders described by the rectangles Ce, ao, ...; for, obviously, the difference of the hemisphere and the sum of these cylinders lying entirely within it is less than the difference between the sum of these same cylinders and the sum of the cylinders described by the rectangles Cf, av, bx, ... which lie partly outside the hemisphere. But, from the Fig. it is evident that the difference of the cylinders described by Ce, ao, ... and those described by Cf, av, ... is the cylinder described by Cf; since, cylinder described by Ce = cylinder described by av; cylinder described by ao = cylinder described by bx (163, *Cor.*) . . . . . .

Hence, the difference of the hemisphere and the sum of the cylinders described by the rectangles Ce, ao, ... being less than the cylinder described by Cf, as the latter may be made less than any given solid by increasing n indefinitely, the hemisphere is the limit of the sum of the cylinders described by Ce, ao, ...

By reasoning in the same way it may be shown that the cone described by the triangle CAB is the limit of the sum of the cylinders described by the rectangles Ct, as, ...

Therefore the sum of the volumes of the cylinders described by Cf, ag, ... , that is, the volume of the cylinder described by the figure AD is equal to the sum of the volumes of the cone and hemisphere. But the volume of

the cone is one-third of the volume of the cylinder (172, i., iii.); hence, the volume of the hemisphere is two-thirds of the volume of its circumscribed cylinder.

As the same proof applies to the other hemisphere and its circumscribed cylinder, hence

The volume of the sphere is two-thirds of the volume of its circumscribed equilateral cylinder.

Cor. i. If the diameter of a sphere be called D, and the area of one of its great circles S; then

$$\text{volume of circumscribed cylinder} = S \times D = \frac{\pi D^3}{4}$$

(168, *Cor.* i., 110);

therefore,

$$\text{volume of sphere} = \tfrac{2}{3}\, S \times D = \tfrac{1}{6}\, \pi D^3,$$

the same result as that already obtained (175).

Cor. ii. A cone standing on the same base with the circumscribed cylinder, and having the immoveable diameter for its axis, has a volume which is one half that of the sphere (172, *Cor.* iii.); hence

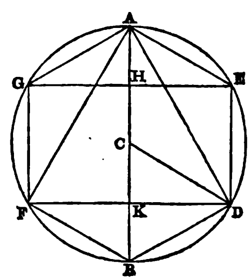
The volumes of the cone, sphere, and cylinder are related to each other as the numbers 1, 2, 3.

Theorem.

177. *The volume of a sphere is related to the volume of an inscribed equilateral cone as* 32 : 9.

The equilateral triangle ADF (Fig. 151) revolving round the fixed diameter AB will generate an equilateral cone, and the circle AEB will generate a sphere circumscribed about it.

Fig. 151.



Let r be the radius of the sphere, and $r_{\prime}$ the radius of the base of the cone; then

$$\text{volume of sphere} = \tfrac{4}{3}\, \pi r^3 \ (175);$$

and

$$\text{volume of cone} = \tfrac{1}{3}\, \pi r_{\prime}^2 \times AK$$

(172, *Cor.* iii.).

But

$$r_,^2 = r^2 - \left(\frac{r}{2}\right)^2 = \tfrac{3}{4} r^2 \text{ (90 } Cor. \text{ i. 74)};$$

and

$$AK = AC + CK = \tfrac{3}{2} r.$$

Hence,

$$\text{volume of cone} = \tfrac{1}{3}\pi \times \tfrac{3}{4} r^2 \times \tfrac{3}{2} r = \tfrac{3}{8}\pi r^3,$$

and

$$\text{volume of sphere : volume of cone} :: \tfrac{4}{3}\pi r^3 : \tfrac{3}{8}\pi r^3 :: 32 : 9.$$

THEOREM.

178. *The volume of a sphere is related to the volume of an inscribed hexagonal spheroid as* 4 : 3.

The regular hexagon AEDBFG (Fig. 151) revolving round the fixed diameter AB, which passes through the vertices of two of the angles, will generate a hexagonal spheroid consisting of two equal cones described by the right-angled triangles AHE, BKD, and a cylinder described by the rectangle HD.

Let r be the radius of the sphere, and $r_,$ the radius of the common bases of the cylinder and cones; then

$$\text{volume of sphere} = \tfrac{4}{3}\pi r^3 \text{ (175)};$$

and

$$\text{volume of spheroid} = \tfrac{2}{3}\pi r_,^2 \times AH + \pi r_,^2 \times r$$

(172, *Cor.* iii., 168, *Cor.* i.).

But

$$r_,^2 = r^2 - \left(\frac{r}{2}\right)^2 = \tfrac{3}{4} r^2 \text{ (90, } Cor. \text{ i., 74)};$$

and

$$AH = BK = \frac{r}{2}.$$

Hence,

$$\text{volume of spheroid} = \tfrac{2}{3}\pi \times \tfrac{3}{4} r^2 \times \frac{r}{2} + \tfrac{3}{4}\pi r^3 = \pi r^3;$$

and

$$\text{vol. of sphere : vol. of spheroid} :: \tfrac{4}{3}\pi r^3 : \pi r^3 :: 4 : 3.$$

SECTION III.

SIMILAR SOLIDS.

179. *Def.* Two polyhedrons are *similar* when they have their solid angles respectively equal, and are bounded by the same number of plane figures, similar and similarly situated, two and two.

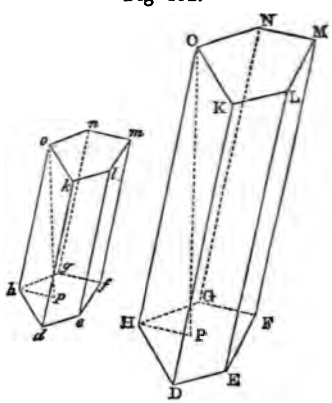
The equal angles in two similar polyhedrons are called *homologous* angles; and lines joining the vertices of homologous angles are called homologous lines.

THEOREM.

180. *The surfaces of similar prisms are related to each other as the squares of their homologous dimensions.*

Let DEFGHO and *defgho* (Fig. 152) be two similar prisms standing on the bases B and *b*, respectively; and from the vertices of the homologous angles O, *o*, let fall on these bases the perpendiculars OP, *op*. Put OP = A, *op* = *a*, and draw HP, *hp*.

Fig. 152.



The right-angled triangles OPH, *oph*, are similar; for, since the homologous angles H, *h*, are equal, the plane angles and edges of *h* must coincide respectively with those of H, when the solid angle *h* is properly applied to H; and, consequently, the edges HO, and *ho*, are equally inclined to the respective bases B and *b*. Therefore

A : *a* : : HO : *ho* : : HD : *hd* : : DE : *de* (86, 94,) . . . . (1).

Let T and *t* be the total surfaces of the two prisms, and

let the lateral parallelograms OD, KE, LF, ... be represented by S, S′, S″, ... and *od*, *ke*, *lf*, ... by *s*, *s*′, *s*″. . . . . Then, since the faces two and two are similar (179),

$$S : s :: HD^2 : hd^2 \ (117)$$

$$S' : s' :: DE^2 : de^2$$

$$S'' : s'' :: EF^2 : ef^2,$$

and so of the others.

Also

$$B : b :: HD^2 : hd^2 \ (117),$$

and

$$2B : 2b :: HD^2 : hd^2.$$

But, from proportion (1), the second ratios in all these proportions are equal; therefore

$$S + S' + S'', \&c. : s + s' + s'', \&c. :: S : s :: HD^2 : hd^2$$

and

$$S + S' + S'' \ldots + 2B : s + s' + s'' \ldots + 2b :: HD^2 : hd^2;$$

that is,

$$T : t :: HD^2 : hd^2 :: HO^2 : ho^2 :: A^2 : a^2.$$

Hence the lateral and also the total surfaces of similar prisms are related to each other as the squares of their homologous dimensions.

Cor. i. A cylinder being a prism with a circular base (155, VII.),

The lateral and also the total surfaces of similar cylinders are related to each other as the squares of their axes—of their altitudes—of the diameters of their bases—of any other homologous dimensions.

THEOREM.

181. *The volumes of similar prisms are related to each other as the cubes of their homologous dimensions.*

Let V and v be the respective volumes of the similar prisms DEFGHO and *defgho* (Fig. 152) whose bases are B, b, and altitudes A, a. Then

$$V = B \times A \ (168), \quad v = b \times a.$$

Therefore

$$V : v :: B \times A : b \times a \ \ldots\ldots (1).$$

But

$$B : b :: HD^2 : hd^2 :: HO^2 : ho^2 :: A^2 : a^2 \ (117) \ldots (2);$$

consequently, from proportions (1) and (2),

$$V : v :: A^3 : a^3 :: HO^3 : ho^3 :: HD^3 : hd^3 \ \ldots\ldots$$

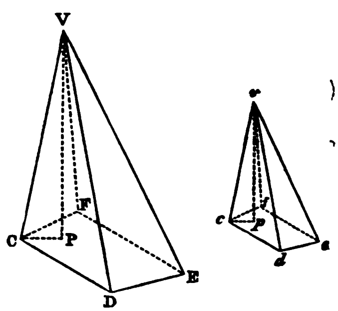
Cor. The volumes of similar cylinders are related to each other as the cubes of their altitudes—of their axes—of the radii and diameters of their bases .... (155, VII.).

THEOREM.

182. *The lateral and also the total surfaces of similar pyramids are related to each other as the squares of their homologous dimensions.*

Let CDEFV and *cdefv* (Fig. 152) be two similar pyramids standing on the bases B and b respectively. Let fall the perpendiculars VP = A, and $vp = a$; and draw CP, cp.

Fig. 153.



The two right-angled triangles VPC, vpc, are similar; for the homologous solid angles C and c being equal, the plane angles and edges of c must coincide respectively with the plane angles and edges of C when the two solid angles are properly applied one to the other. Therefore, the edges VC and vc are equally in-

clined to the bases; that is, the angle VCP $=vcp$. Consequently

$$A : a :: VC : vc :: CD : cd \ldots . (1).$$

Let T and t be the total surfaces of the two pyramids, and let the areas of the lateral triangles VCD, VDE, VEF, ... be represented by S, S′, S″, ..., and the areas of vcd, vde, vef, ... by s, s', s'', ... Then, since the faces, two and two, are similar (179),

$$S : s :: CD^2 : cd^2 \ (117),$$

$$S' : s' :: DE^2 : de^2,$$

$$S'' : s'' :: EF^2 : ef^2,$$

and so of the others.
Also

$$B : b :: CD^2 : cd^2 \ldots .$$

But, from proportion (1), the second ratios in these proportions are equal; therefore

$$S + S' + S'', \&c., : s + s' + s'', \&c., :: CD^2 : cd^2$$

and

$$S + S' + S'' \ldots + B : s + s' + s'' \ldots + b :: CD^2 : cd^2;$$

that is,

$$T : t :: CD^2 : cd^2 :: CV^2 : cv^2 :: A^2 : a^2 \ldots .$$

Cor. A cone being a pyramid with a circular base (155, XI.), hence

The lateral and also the total surfaces of similar cones are related to each other as the squares of their altitudes—of their axes—of the radii and diameters of their bases, &c.

THEOREM.

183. *The volumes of similar pyramids are related to each other as the cubes of their homologous dimensions.*

Let V and v be the respective volumes of the two similar pyramids CDEFV and $cdefv$ (Fig. 153). Then (172)

$$V = \frac{B \times A}{3}, v = \frac{b \times a}{3};$$

and

$$V : v : : B \times A : b \times a.$$

But

$$B : b : : CD^2 : cd^2 : : CV^2 : cv^2 : : A^2 : a^2 \ldots ;$$

Therefore

$$V : v : : A^3 : a^3 : : CV^3 : cv^3. \ldots$$

Cor. The volumes of similar cones are related to each other as the cubes of their altitudes—of their axes—of the radii and diameters of their bases, &c.

THEOREM.

184. *The surfaces of spheres are related to each other as the squares of their radii—of their diameters—of any of their homologous lines.*

Let S and s be the convex surfaces of two spheres whose diameters are D, d, and radii R, r, respectively. Then

$$S = 4\pi R^2 \text{ (159)}$$

and

$$s = 4\pi r^2.$$

Therefore

$$S : s : : 4\pi R^2 : 4\pi r^2 : : R^2 : r^2 : : D^2 : d^2 \ldots$$

The surface of a sphere whose diameter is one foot being 3·14159 square feet (159, *Cor.* i.), the surface of any other

sphere is obtained by multiplying 3·14159 by the second power of the number which expresses in feet the length of its diameter.

Thus, if D = 10 feet,

$$\text{surface of sphere} = 3{\cdot}14159 \times 100 = 314{\cdot}159 \text{ square feet.}$$

THEOREM.

185. *The volumes of spheres are related to each other as the cubes of their diameters—of their radii—of any of their homologous lines.*

Let V and v be the volumes of two spheres whose diameters are D, d, and radii R, r, respectively.

Then

$$V = \tfrac{4}{3}\pi R^3,\ v = \tfrac{4}{3}\pi r^3 \ (175);$$

therefore

$$V : v :: \tfrac{4}{3}\pi R^3 : \tfrac{4}{3}\pi r^3 :: R^3 : r^3 :: D^3 : d^3 \ . \ . \ . \ .$$

The volume of a sphere whose diameter is 1 being ·5236 (175), the volume of any other sphere will be obtained by multiplying ·5236 by the third power of the number which expresses the length of its diameter.

Thus the volume of a sphere whose diameter is 5 feet is

$$\cdot5236 \times 5^3 = 65{\cdot}45 \text{ cubic feet.}$$

186. When solids of revolution are generated by similar plane figures revolving round homologous dimensions of the figures as axes, the solids are similar. Hence

Two equilateral cones, one inscribed in a sphere, the other circumscribed about the same sphere, have

their surfaces as 1 : 4,

and

their volumes as 1 : 8 (74);

inscribed and circumscribed equilateral cylinders have

their surfaces as $1 : 2$,

and

their volumes as $1 : \sqrt{8} = 2\sqrt{2}$;

inscribed and circumscribed regular hexagonal spheroids have

their surfaces as $3 : 4$,

and

their volumes as $3\sqrt{3} : 8$.

Exercises.

1. The four diagonals of a parallelopiped pass through the same point and are bisected in it.

The point is called the *centre* of the parallelopiped.

2. In a solid formed by six planes parallel, two and two, the opposite faces are equal parallelograms.

3. Four points which are not in the same plane being given, one and only one sphere can be drawn so that each of the points shall lie in its surface.

How many spheres can be described about a given tetrahedron?

4. The intersection of the surfaces of two spheres is the circumference of a circle having its centre in the line joining the centres of the spheres and its plane at right angles to the same line.

5. What is the volume of a cube whose diagonal measures 300 feet?

6. How many square miles in the surface of the earth, the circumference being 25000 miles?

7. The surface of a sphere is related to the surface of an inscribed regular hexagonal spheroid as $2 : \sqrt{3}$.

8. When a triangle revolves round one of its sides as an axis, the volume of the solid which it generates is measured by one-third of the product of its area and the circumference of a circle of which the perpendicular from the opposite angle is the radius.

9. The surface of a right cylinder circumscribed about a sphere is a mean proportional between the surfaces of the sphere and of a circumscribed equilateral cone; and its volume is a mean proportional between the volumes of the sphere and cone.

Prove that the same relation exists between the sphere and the inscribed cylinder and cone.

In solving some of the problems which follow, it will be necessary to remember that a gallon of water weighs 10 lbs., and a cubic foot of water weighs 62·5 lbs., or 1000 ozs., nearly.

10. If limestone be 2·6 times heavier than water, what will be the weight of a block of it which measures 2 feet in length, 9 inches in width, and 15 inches in depth?

11. How many gallons of water can a cistern hold whose length is 10 feet, breadth 6 feet, and height 5 feet?

12. Show that a regular tetrahedron the edges of which are each 24 inches long contains ·9427 cubic feet.

13. A frustum of a cone is 31 feet in height and the radius of one end is 10 feet; required the radius of the other end, the volume of the frustum being equivalent to that of a cylinder the radius of whose base is 62 feet, and height 4 inches.

14. What is the length of the radius of a sphere whose convex surface contains 16 square feet?

15. A cask whose height is 4 feet measures 12 feet in circumference in the middle, and 10 feet at either end; how many gallons can it contain, the cask being regarded as two truncated cones with a common base?

16. If the materials of which the earth is composed be taken, on the average, as 5·5 times heavier than water, what will be the weight of the earth, its diameter being 8000 miles?

17. Cast iron is about 7·25 heavier than water; required the weight of a hollow shell 12 inches in diameter, the thickness of the iron being 2 inches.

18. The sum of the squares of the four diagonals of a parallelopiped is equal to the sum of the squares of its edges.

19. Two segments 15 and 16 inches in height respectively, are cut from a sphere whose diameter is 5 feet; required the convex surface of the part that remains.

20. A piece of timber is 12 feet long and has square ends; a side of one of the squares measures 15 inches, and a side of the other measures 6 inches; how many cubic feet does it contain?

In treating of loci in page 85, the point was supposed to be confined to one plane; when this restriction is removed the point will trace out a line or surface, and these are called *the locus of the point in space.*

Thus, whereas the locus of a point moving in one plane at a given distance from a fixed point in the plane is the circumference of a circle, the locus of a point in space at a given distance from a fixed point is the surface of a sphere of which the given distance is the radius and the fixed point the centre. So also the locus of a point in a plane equidistant from two given points in that plane is a *line* which bisects perpendicularly the line joining the two given points; but the locus of a point in space equidistant from two given points is a *plane* bisecting perpendicularly the line which joins the two given points.

Required the locus of a point equidistant from the vertices of the three angles of a triangle.

Required the locus of a point equidistant from two given lines,

1°. When the lines are parallel;

2°. When the lines intersect.

Required the locus of a point equidistant from two given planes.

THE END.